Transfer function

AE353

Spring 2025

Bretl

WHAT WE SAW

If the desired wheel angle is a sine wave with frequency ω

$$q_{des}(t) = a \sin(\omega t)$$

Then the actual wheel angle is also a sine wave with the same frequency ω and with magnitude and angle that depend on ω

$$q(t) = \underbrace{(\text{some transient response})}_{\text{decayed to zero}} + m\, a \sin(\omega t + \theta)$$

EXAMPLES

$\omega = (2\pi/1)$ $m = 0.678$ $\theta = -2.64$

$\omega = (2\pi/2)$ $m = 1.79$ $\theta = -0.723$

$\omega = (2\pi/5)$ $m = 1.09$ $\theta = -0.162$

slower

INPUT

$$q_{des}(t) = a \sin(\omega t)$$

$$\dot{x} = Ax + Bu$$ dynamic model

$$y = Cx$$ sensor model

$$u = -K(\hat{x} - x_{des})$$ controller (w/ tracking)

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$$ observer

OUTPUT

$$q(t) = m\, a \sin(\omega t + \theta) + (\ldots)$$

MODEL (1)

$\dot{x} = Ax + Bu$ dynamic model

$y = Cx$ sensor model

$u = -K(\hat{x} - x_{des})$ controller (w/ tracking)

$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$ observer

$$\dot{x} = Ax + Bu = Ax + B(-K(\hat{x} - x_{des}))$$

$$= Ax - BK\hat{x} + BKx_{des}$$

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y) = A\hat{x} + B(-K(\hat{x} - x_{des})) - L(C\hat{x} - Cx)$$

$$= LCx + (A - BK - LC)\hat{x} + BKx_{des}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} BK \\ BK \end{bmatrix} x_{des}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} BK \\ BK \end{bmatrix} x_{des}$$

MODEL (2)

the state to track: $x_{des} = \begin{bmatrix} q_{des} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1} [q_{des} - \cancel{q_e}^{\,0}]$

INPUT: $e_1^T x_{des} =$

OUTPUT: $e_1^T x = [1 \quad 0] \begin{bmatrix} q - \cancel{q_e}^{\,0} \\ v - v_e \end{bmatrix} = [q]$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \overbrace{\begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix}}^{A_m} \overset{x_m}{\begin{bmatrix} x \\ \hat{x} \end{bmatrix}} + \overbrace{\begin{bmatrix} BKe_1 \\ BKe_1 \end{bmatrix}}^{B_m} \overset{u_m}{[q_{des}]}$$

$$\underbrace{[\, q \,]}_{y_m} = \underbrace{[\, e_1^T \quad 0 \,]}_{C_m} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \underbrace{[\, 0 \,]}_{D_m} [q_{des}]$$

$\dot{x}_m = A_m x_m + B_m u_m$

$y_m = C_m x_m + D_m u_m$

General Result

single input

$$\dot{x}_m = A_m x_m + B_m u_m$$

$$y_m = C_m x_m + D_m u_m$$

single output

transient (decays to zero)

$$u_m(t) = \sin(\omega t) \implies y_m(t) = (\cdots) + |H(j\omega)| \sin(\omega t + \angle H(j\omega))$$

$$u_m(t) = \cos(\omega t) \implies y_m(t) = (\cdots) + \underbrace{|H(j\omega)|}_{\text{magnitude}} \cos(\omega t + \underbrace{\angle H(j\omega)}_{\text{angle}})$$

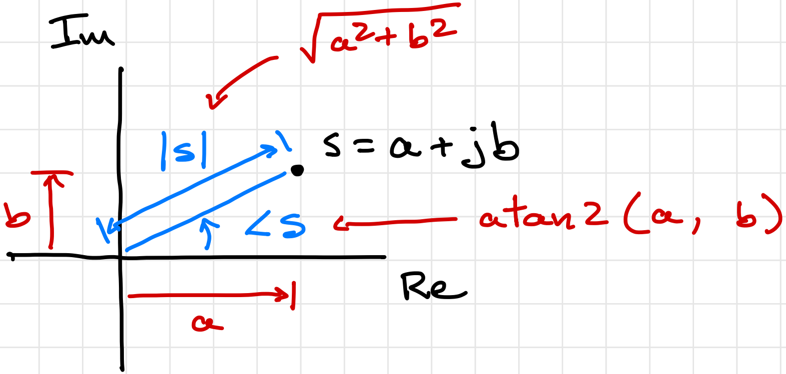
a complex number

$$\underbrace{H(s)}_{\text{another complex number}} = C_m(sI - A_m)^{-1} B_m + D_m$$

← TRANSFER FUNCTION

WHY???

COMPLEX NUMBERS



$$s = |s| e^{j\angle s} = |s|\left(\cos(\angle s) + j\sin(\angle s)\right)$$

$$\frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) = \frac{1}{2j}\left((\cos(\omega t) + j\sin(\omega t)) - (\overbrace{\cos(-\omega t)}^{\cos(\omega t)} + j\overbrace{\sin(-\omega t)}^{-\sin(\omega t)})\right)$$

$$= \frac{1}{2j}\left(\cancel{\cos(\omega t)} + j\sin(\omega t) - \cancel{\cos(\omega t)} + j\sin(\omega t)\right)$$

$$= \frac{1}{2j}\left(2j\sin(\omega t)\right)$$

$$= \sin(\omega t)$$

$$\dot{x}_m = A_m x_m + B_m u_m$$
$$y_m = C_m x_m + D_m u_m$$

WHY?

$\downarrow$ video on solution to systems with input

$$y_m(t) = C_m e^{A_m t} x_m(0) + \int_0^t e^{A_m(t-\tau)} B_m u_m(\tau)\, d\tau$$

$\downarrow$ video on deriving the transfer function

IF...

$$u_m(t) = e^{st}$$

some complex number

THEN...

$$y_m(t) = \underbrace{C_m e^{A_m t}\left(x_m(0) - (sI - A_m)^{-1} B_m\right)}_{\text{transient}} + \underbrace{C_m (sI - A_m)^{-1} B_m e^{st}}_{\text{steady-state}}$$

$\downarrow$ video on deriving the frequency response

$$u_m(t) = \sin(\omega t) \quad \Rightarrow \quad y_m(t) = (\cdots) + \frac{1}{2j}\left(H(j\omega) e^{j\omega t} - H(j\omega) e^{-j\omega t}\right)$$

$$= \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \qquad = |H(j\omega)| \sin\left(\omega t + \angle H(j\omega)\right)$$

BODE PLOTS

$\boxed{|H(j\omega)| \text{ as a function of } \omega}$ ← both axes on log scale

$\boxed{\angle H(j\omega) \text{ as a function of } \omega}$ ← ω on log scale, $\angle H(j\omega)$ on linear scale

STRUCTURE

$$H(s) = k\,\frac{n(s)}{d(s)}$$

$$|H(s)| = |k|\,\frac{|n(s)|}{|d(s)|} \Rightarrow \log|H| = \log|k| + \log|n(s)| - \log|d(s)|$$

$$\angle H(s) = \angle k + \angle n(s) - \angle d(s)$$

↑

the Bode plot is the sum and difference of a bunch of simple plots

Converting to/from "decibels" (dB)

absolute $\qquad\qquad$ dB

$$m \longrightarrow 20 \log_{10} m$$

$$10^{\left(\tilde{m}/20\right)} \longleftarrow \tilde{m}$$

BANDWIDTH

The frequency ω at which $|H(j\omega)|$ drops below -3 dB.