# Observers (Part 3)

AE353

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state

input

$$\dot{x} = Ax + Bu$$ ← dynamic model

$$y = Cx$$ ← sensor model

output

$$u = -K\hat{x}$$ ← controller

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$$ ← observer

DOES THE CONTROLLER STILL WORK?

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

sensor model

dynamic model

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$$

observer

$$u = -K\hat{x}$$

controller

- does this screw anything up?
- must we change our choice of $K$?

$$\dot{x} = (A-BK)x$$

is NOT the closed-loop system anymore!

## DOES THE CONTROLLER STILL WORK?

$$\dot{x} = Ax + Bu \qquad \dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y) \qquad u = -K\hat{x}$$
$$y = Cx$$

dynamic model  sensor model  observer  controller

We had found

$$\dot{x}_{err} = (A - LC)\,x_{err} \qquad \text{where} \qquad x_{err} = \hat{x} - x.$$

$$\hat{x} = x_{err} + x$$

Now, let's find

$$\dot{x} = Ax + Bu = Ax - BK\hat{x} = Ax - BK(x_{err} + x)$$
$$= Ax - BKx_{err} - BKx = (A - BK)x - BKx_{err}$$

Write in matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{err} \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ x_{err} \end{bmatrix}$$

# DOES THE CONTROLLER STILL WORK?

$$\dot{x} = Ax + Bu \qquad \dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y) \qquad u = -K\hat{x}$$
$$y = Cx$$

dynamic model — sensor model — observer — controller

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{err} \end{bmatrix} = \underbrace{\begin{bmatrix} A-BK & -BK \\ 0 & A-LC \end{bmatrix}}_{\text{stability depends on the eigenvalues of this matrix}} \begin{bmatrix} x \\ x_{err} \end{bmatrix}$$ ← closed-loop system

FACT:

$$\det\left(\begin{bmatrix} F & H \\ 0 & G \end{bmatrix}\right) = \det(F)\det(G)$$

CONSEQUENCE:

the eigs of $\begin{bmatrix} A-BK & -BK \\ 0 & A-LC \end{bmatrix}$

are the <u>union</u> of the eigs of $A-BK$ and of $A-LC$

so... choose $K$ <u>as if</u> $\dot{x} = (A-BK)x$ <u>were</u> the closed-loop system — this is still OK!

# SEPARATION PRINCIPLE

you can design the observer and the controller separately

1. design the observer while ignoring the controller (works for arbitrary $u$)

2. design the controller assuming the state estimate is perfect

HOW TO PREDICT $x(t)$ AND $x_{err}(t)$ ?

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{x}_{err} \end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix} A-BK & -BK \\ 0 & A-LC \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} x \\ x_{err} \end{bmatrix}}_{z}$$

$$z(t) = e^{Ft} z(0) \qquad z(t) = e^{F(t-t_0)} z(t_0)$$

CAN I SOLVE FOR $x$ AND $\hat{x}$ INSTEAD OF $x$ AND $x_{err}$? ← YES

$$\dot{x} = Ax + Bu \qquad\qquad \dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y) \qquad\qquad u = -K\hat{x}$$

$$y = Cx$$

$$\dot{x} = Ax + Bu$$

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$$

start with this, then plug in expressions for $u$ and $y$, then collect terms in $x$ and $\hat{x}$, and finally write the result in matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} \qquad\qquad & \qquad\qquad \\ & \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

HOW DO I LINEARIZE A NONLINEAR SENSOR MODEL?

$$\underbrace{\dot{m}}_{\dot{x}} = f(m,n) \approx \underbrace{\cancel{f(m_e, n_e)}}_{0} + \underbrace{\left.\frac{\partial f}{\partial m}\right|_{(m_e,n_e)}}_{A} \underbrace{(m-m_e)}_{x} + \underbrace{\left.\frac{\partial f}{\partial n}\right|_{(m_e,n_e)}}_{B} \underbrace{(n-n_e)}_{u}$$

$$o = g(m,n) \approx g(m_e, n_e) + \left.\frac{\partial g}{\partial m}\right|_{(m_e,n_e)} (m-m_e) + \left.\frac{\partial g}{\partial n}\right|_{(m_e,n_e)} (n-n_e)$$

$$\Updownarrow$$

$$\underbrace{o - g(m_e,n_e)}_{y} \approx \underbrace{\left.\frac{\partial g}{\partial m}\right|_{(m_e,n_e)}}_{C} \underbrace{(m-m_e)}_{x} + \underbrace{\left.\frac{\partial g}{\partial n}\right|_{(m_e,n_e)}}_{D} \underbrace{(n-n_e)}_{u}$$

ALSO SEE REFERENCE PAGE (STATE ESTIMATION)