

Ackermann's method (part 1)

AE 353

Spring 2025

Bret 1

A, B

← description of state-space model

p_1, \dots, p_n

← list of desired eigenvalue locations

number
of states

EIGENVALUE PLACEMENT

K

← gain matrix for which $A-BK$
has eigenvalues at desired locations

WHAT CHARACTERISTIC POLYNOMIAL HAS ROOTS AT p_1, \dots, p_n ?

$$(s - p_1) = s - p_1$$

$$(s - p_1)(s - p_2) = s^2 - (p_1 + p_2)s + p_1 p_2$$

$$(s - p_1)(s - p_2)(s - p_3) = s^3 - (p_1 + p_2 + p_3)s^2 + (p_1 p_2 + p_2 p_3 + p_3 p_1)s - p_1 p_2 p_3$$

⋮

$$= s^n + r_1 s^{n-1} + r_2 s^{n-2} + \dots + r_{n-1} s + r_n$$

It is easy to compute the coefficients

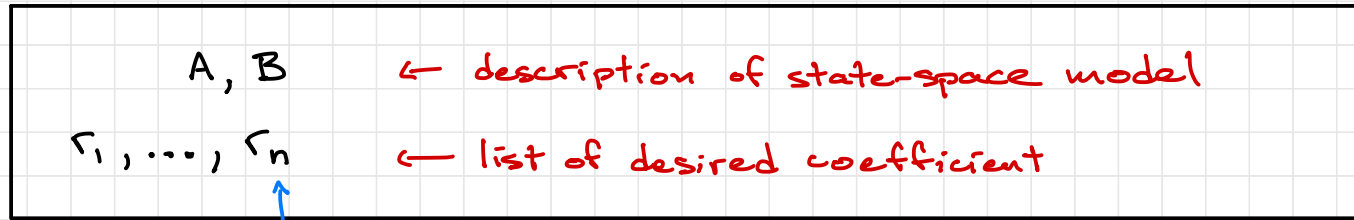
r_1, \dots, r_n

given the eigenvalue locations

p_1, \dots, p_n .

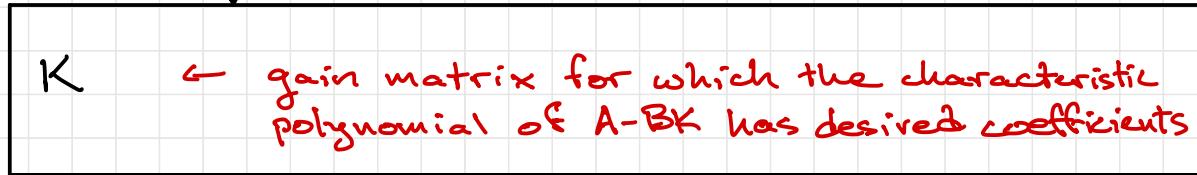
NUMERIC !!!

↑
np.poly(p)



number of states

EIGENVALUE PLACEMENT



STRATEGY

- ① Find K in special case when it is easy
- ② Transform general case to special case

IF:

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$K = [k_1 \quad k_2]$$

THEN:

$$A - BK = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 \\ 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(sI - (A - BK)) &= \det \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 \\ 1 & 0 \end{bmatrix} \right) = \det \left(\begin{bmatrix} s + (a_1 + k_1) & a_2 + k_2 \\ -1 & s \end{bmatrix} \right) \\ &= s^2 + (a_1 + k_1)s + (a_2 + k_2) \end{aligned}$$

SO:

if you want

$$s^2 + r_1 s + r_2$$

then

$$k_1 = r_1 - a_1$$

$$k_2 = r_2 - a_2$$

← easy! no symbolic computation!

Controllable Canonical Form (CCF)

$$A = \begin{bmatrix} [-a_1 & \dots & -a_n] \\ [I_{(n-1) \times (n-1)}] [0_{(n-1) \times 1}] \end{bmatrix} \quad B = \begin{bmatrix} [1] \\ [0_{(n-1) \times 1}] \end{bmatrix}$$

Facts

$$\det(sI - A) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$$

$$A - BK = \begin{bmatrix} [-a_1 - k_1 & \dots & -a_n - k_n] \\ [I] [0] \end{bmatrix}$$

$$\det(sI - (A - BK)) = s^n + (a_1 + k_1) s^{n-1} + \dots + (a_{n-1} + k_{n-1}) s + (a_n + k_n)$$

Consequence

if you want

$$s^n + r_1 s^{n-1} + \dots + r_{n-1} s + r_n$$

then

$$k_1 = r_1 - a_1 \quad \dots \quad k_n = r_n - a_n$$

*no symbolic
computation!*

If we could put a system in CCF...

$$\dot{x} = Ax + Bu$$

$$\downarrow \quad x = Vz$$

$$\dot{z} = A_{ccf}z + B_{ccf}u$$

$$V\dot{z} = AVz + Bu$$

$$\dot{z} = \boxed{V^{-1}AV}z + \boxed{V^{-1}B}u$$

$A_{ccf} \qquad B_{ccf}$

Then ...

$$u = -\overbrace{K_{ccf}}^{\text{easy to find}} z = -K_{ccf}V^{-1}x$$
$$= -\underbrace{(K_{ccf}V^{-1})}_K x$$

$K \text{ (what we want)}$

How to find A_{ccf} ? (B_{ccf} is always the same)

eigenvalues are
invariant to
coordinate transformation

$$\det(sI - A_{ccf}) = \det(sI - V^{-1}AV)$$

$$= \det(sV^{-1}V - V^{-1}AV)$$

$$= \det(V^{-1}(sI - A)V)$$

$$= \det(V^{-1}) \det(sI - A) \det(V)$$

$$= \cancel{\det(V^{-1})} \det(sI - A) \cancel{\det(V)}$$

$$= \det(sI - A)$$

$$\leftarrow V^{-1}V = I$$

$$\leftarrow \det(MN) = \det(M) \det(N)$$

$$\leftarrow \det(M^{-1}) = \det(M)^{-1}$$



a_1, \dots, a_n are the coefficients of the
characteristic polynomial of A

↑ np.poly(A)