Diagonalization (Part 2)

AE353

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Bretl

LAST TIME

$$\dot{x} = Ax + Bu$$ ← state-space model (dynamics)

$$u = -Kx$$ ← linear state feedback (controller)

$$\downarrow$$

$$\dot{x} = (A-BK)x$$ ← closed-loop system

$$\downarrow$$

$$x(t) = e^{(A-BK)t}\,x(0)$$ ← solution (by matrix exponential)

$x(t) \to 0$ as $t \to \infty$ if and only if all eigenvalues of $A-BK$ have negative real part } asymptotic stability

↑

OUR GOAL IS TO PROVE THIS RESULT

$\dot{x} = Fx$ ← for which F does $x(t) \to 0$ as $t \to \infty$???

STRATEGY

1. Answer this question in the special case when F is diagonal
2. Show how to rewrite (almost) any F as diagonal
3. Answer this question for (almost) any F

① Answer the question in the special case when F is diagonal

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \Rightarrow e^{Ft} = I + Ft + \frac{1}{2!}(Ft)^2 + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2!}(s_1 t)^2 & 0 \\ 0 & \frac{1}{2!}(s_2 t)^2 \end{bmatrix} + \dots$$

$$x(t) = e^{Ft} x(0)$$

$$= \begin{bmatrix} 1 + s_1 t + \frac{1}{2!}(s_1 t)^2 + \dots & 0 \\ 0 & 1 + s_2 t + \frac{1}{2!}(s_2 t)^2 + \dots \end{bmatrix}$$

$$\begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix}\begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix}$$

$$= \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}$$

← when F is diagonal, e^{Ft} is easy to find

WHAT ARE THE TERMS ?

if $s = \sigma + j\omega$ then $e^{st} = e^{(\sigma + j\omega)t} = e^{(\sigma t + j\omega t)}$

$$= e^{\sigma t} e^{j\omega t} = e^{\sigma t}\left(\cos(\omega t) + j \sin(\omega t)\right)$$

② Show how to rewrite (almost) any F as diagonal (1/3)

KEY: COORDINATE TRANSFORMATION

old coordinates   new coordinates

$$x = Vz$$

EXAMPLE →

$$\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ q \end{bmatrix}$$

EXAMPLE →

$$\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} q+v \\ q-v \end{bmatrix}$$

② Show how to rewrite (almost) any F as diagonal (2/3)

solve for x(t) with matrix exponential

$$\dot{x} = Fx$$

plug in $\boxed{x = Vz}$ for some invertible V

$z = V^{-1}x$

$$V\dot{z} = FVz$$

solve for $\dot{z}$

$$\dot{z} = (V^{-1}FV)z$$

solve for z(t) with matrix exponential

$$z(t) = e^{(V^{-1}FV)t} z(0)$$

plug in $z = V^{-1}x$ and solve for x

$$V^{-1}x(t) = e^{(V^{-1}FV)t} V^{-1}x(0)$$

$$\left.\begin{array}{l} x(t) = Ve^{(V^{-1}FV)t} V^{-1}x(0) \\ \\ x(t) = e^{Ft} x(0) \end{array}\right\}$$

must be equal for any invertible V

② Show how to rewrite (almost) any F as diagonal (3/3)

$$x(t) = e^{Ft} x(0)$$

$$= V \underbrace{e^{(V^{-1}FV)t}}_{\text{choose } V \text{ so } V^{-1}FV \text{ is diagonal}} V^{-1} x(0)$$ for <u>any</u> invertible V

$$V^{-1}FV = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \qquad V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$\Rightarrow \quad FV = V \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

$$F \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

$$\begin{bmatrix} Fv_1 & Fv_2 \end{bmatrix} = \begin{bmatrix} v_1 s_1 & v_2 s_2 \end{bmatrix}$$

$$\boxed{\begin{aligned} Fv_1 &= v_1 s_1 \\ Fv_2 &= v_2 s_2 \end{aligned}}$$

v_1 and v_2 are eigenvectors of F

s_1 and s_2 are eigenvalues of F

③ Answer the question for (almost) any F

$\boxed{\dot{x} = Fx \quad \leftarrow \text{for which } F \text{ does } x(t) \to 0 \text{ as } t \to \infty \text{ ???}}$

We have shown:

$$x(t) = e^{Ft}x(0) = Ve^{(V^{-1}FV)t}V^{-1}x(0)$$

This means that

$$e^{Ft} \to 0 \quad \text{if and only if} \quad e^{(V^{-1}FV)t} \to 0$$

for some invertible V. If we choose V as a matrix with the eigenvectors of F as its columns (and if all eigenvalues of F are distinct) then

$$e^{(V^{-1}FV)t} = e^{\begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}t} = \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}$$

where s_1 and s_2 are the eigenvalues of F (for our 2x2 example).

DEFINITION

The closed-loop system

$$\dot{x} = (A - BK)x$$

is called asymptotically stable if

$$x(t) \to 0 \quad \text{as} \quad t \to \infty$$

for any $x(0)$.

THEOREM

The closed-loop system

$$\dot{x} = (A - BK)x$$

is asymptotically stable if and only if all eigenvalues of

$$A - BK$$

have negative real part.