Diagonalization (Part 1)

AE353

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Bretl

LAST TIME

$$\dot{x} = Ax + Bu$$
$$u = -Kx$$

← state-space model (dynamics)
← linear state feedback (controller)

$$\downarrow$$

$$\dot{x} = (A - BK)x$$

← closed-loop system

$$\downarrow$$

$$x(t) = e^{(A-BK)t} x(0)$$

← solution (by matrix exponential)

$\rightarrow$ $x(t) \to 0$ as $t \to \infty$ if and only if all eigenvalues of $A - BK$ have negative real part

} asymptotic stability

$\uparrow$

OUR GOAL IS TO PROVE THIS RESULT

$\dot{x} = Fx$ ← for which F does $x(t) \to 0$ as $t \to \infty$???

STRATEGY

1. Answer this question in the special case when F is diagonal
2. Show how to rewrite (almost) any F as diagonal
3. Answer this question for (almost) any F

① Answer the question in the special case when F is diagonal

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \Rightarrow e^{Ft} = I + Ft + \frac{1}{2!}(Ft)^2 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2!}(s_1 t)^2 & 0 \\ 0 & \frac{1}{2!}(s_2 t)^2 \end{bmatrix} + \cdots$$

$$x(t) = e^{Ft} x(0)$$

$$= \begin{bmatrix} 1 + s_1 t + \frac{1}{2!}(s_1 t)^2 + \cdots & 0 \\ 0 & 1 + s_2 t + \frac{1}{2!}(s_2 t)^2 + \cdots \end{bmatrix}$$

$$\begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix}\begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix} = \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}$$

← when F is diagonal, e^{Ft} is easy to find

WHAT ARE THE TERMS?

if $s = \sigma + j\omega$ then $e^{st} = e^{(\sigma + j\omega)t} = e^{(\sigma t + j\omega t)}$

$$= e^{\sigma t} e^{j\omega t} = e^{\sigma t}\left(\cos(\omega t) + j \sin(\omega t)\right)$$

(2) Show how to rewrite (almost) any F as diagonal (1/3)

KEY: COORDINATE TRANSFORMATION

old coordinates ↓ $x = V z$ ← new coordinates

EXAMPLE → $\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ q \end{bmatrix}$

EXAMPLE → $\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} q+v \\ q-v \end{bmatrix}$

② Show how to rewrite (almost) any F as diagonal (2/3)

solve for x(t) with matrix exponential

$\dot{x} = Fx$

↓ plug in $x = Vz$ for some invertible V (with $z = V^{-1}x$)

$V\dot{z} = FVz$

↓ solve for $\dot{z}$

$\dot{z} = (V^{-1}FV)z$

↓ solve for $z(t)$ with matrix exponential

$z(t) = e^{(V^{-1}FV)t} z(0)$

↓ plug in $z = V^{-1}x$ and solve for x

$V^{-1}x(t) = e^{(V^{-1}FV)t} V^{-1}x(0)$

$x(t) = Ve^{(V^{-1}FV)t} V^{-1}x(0)$

$x(t) = e^{Ft} x(0)$

must be equal for any invertible V