Linearization

AE353

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Bretl

https://go.aerospace.illinois.edu/ae353-sp25

COURSE WEBSITE

STEP 0 – get EOMs

MODEL OF DYNAMICS

$$\dot{x} = Ax + Bu$$

$$\underbrace{(J + mr^2)}_{c_1}\ddot{q} = \tau - \underbrace{mgr}_{c_2}\sin q$$

STEP 1 – rewrite EOMs as a set of first-order ODEs

$$\ddot{q}$$ ← find time derivative of highest order

$$v = \dot{q}$$ ← define new variables for each time derivative of lower order

$$c_1\dot{v} = \tau - c_2 \sin q$$ ← rewrite EOMs in terms of new variables

$$\dot{q} = v$$ ← add an ODE for each new variable

$$\dot{q} = v$$
$$\dot{v} = (1/c_1)\tau - (c_2/c_1)\sin q$$
← collect ODEs together, solving for time derivatives if necessary

$$\underbrace{\begin{bmatrix}\dot{q}\\ \dot{v}\end{bmatrix}}_{\dot{m}} = \underbrace{\begin{bmatrix} v \\ (1/c_1)\tau - (c_2/c_1)\sin q\end{bmatrix}}_{f(m,n)}$$
← write in standard form

$$m = \begin{bmatrix} q \\ v \end{bmatrix} \qquad n = [\tau]$$

$$\begin{bmatrix} \dot{q}_6 \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ (1/c_1)\tau - (c_2/c_1)\sin q_6 \end{bmatrix}$$

STEP 2 - find an equilibrium point

$$0 = v_e$$

$$0 = (1/c_1)\tau_e - (c_2/c_1)\sin q_{6e}$$

← set time derivatives to zero

$$v_e = 0$$

$$\tau_e = c_2 \sin q_{6e}$$

← solve

$q_{6e} = \pi/2 \quad v_e = 0 \quad \tau_e = c_2$

← pick a solution

$$m_e = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix} \qquad n_e = \begin{bmatrix} c_2 \end{bmatrix}$$

← write in standard form

STEP 3 - define state and input

$$x = \begin{bmatrix} q_6 - (\pi/2) \\ v \end{bmatrix}$$

← $x = m - m_e$

$$u = \begin{bmatrix} \tau - c_2 \end{bmatrix}$$

← $u = n - n_e$

STEP 4 - compute A and B

We defined the state x and the input u like this:

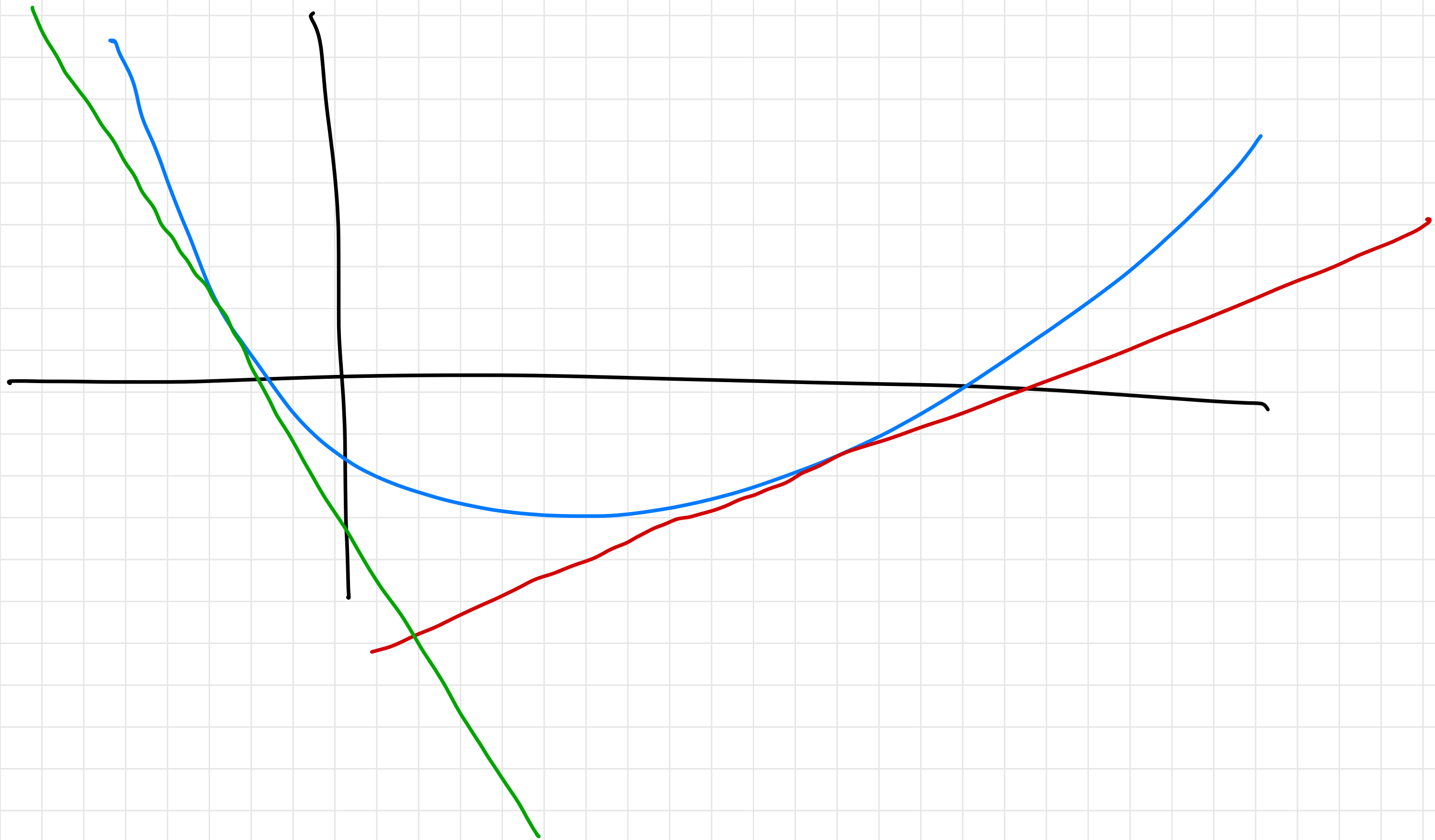
$$x = \begin{bmatrix} q - (\pi/2) \\ v \end{bmatrix} \qquad u = \begin{bmatrix} \tau - c_2 \end{bmatrix}$$

What matrices A and B would make

$$\dot{x} = Ax + Bu \quad \text{and} \quad \begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ (1/c_1)\tau - (c_2/c_1)\sin q \end{bmatrix}$$

describe the same set of ODEs?

$$\underbrace{\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q - (\pi/2) \\ v \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \tau - c_2 \end{bmatrix}}_{u}$$



LINEARIZATION

$$\dot{m} = f(m, n) \approx f(m_e, n_e) + \left.\frac{\partial f}{\partial m}\right|_{(m_e, n_e)} (m - m_e) + \left.\frac{\partial f}{\partial n}\right|_{(m_e, n_e)} (n - n_e)$$

$$\overbrace{\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix}}^{\dot{m}} = \overbrace{\begin{bmatrix} v \\ (1/c_1)\tau - (c_2/c_1)\sin q \end{bmatrix}}^{f(m,n)} \qquad m = \begin{bmatrix} q \\ v \end{bmatrix} \qquad n = [\tau] \qquad m_e = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix} \qquad n_e = [c_2]$$

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$$\approx \begin{bmatrix} v_e \\ (1/c_1)\tau_e - (c_2/c_1)\sin q_e \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ -(c_2/c_1)\cos q_e & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1/c_1 \end{bmatrix}}_{B} u$$

$$\approx \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$x = \begin{bmatrix} q - (\pi/2) \\ v \end{bmatrix} \qquad u = \begin{bmatrix} \tau - 2 \end{bmatrix}$$

MODEL OF CONTROLLER

$$u = -Kx$$

Implement linear state feedback with gain matrix

$$K = \begin{bmatrix} 5 & 1 \end{bmatrix}.$$

$$u = -Kx$$

$$\begin{bmatrix} \tau - 2 \end{bmatrix} = -\begin{bmatrix} 5 & 1 \end{bmatrix} \begin{bmatrix} q - (\pi/2) \\ v \end{bmatrix}$$

$$= \begin{bmatrix} -5(q - (\pi/2)) - v \end{bmatrix}$$

$$\tau = 2 - 5(q - (\pi/2)) - v$$

STATE-SPACE MODEL (linear and time-invariant)

$$\dot{x} = Ax + Bu$$

x: state

u: input

LINEAR STATE FEEDBACK

$$u = -Kx$$

K: gain matrix

EOMs

$\downarrow$ write in standard form

$$\dot{m} = f(m, n)$$

$\downarrow$ choose equilibrium point

$$0 = f(m_e, n_e)$$

$\downarrow$ linearize about equilibrium point

$$x = m - m_e$$

$$u = n - n_e$$

$$A = \left.\frac{\partial f}{\partial m}\right|_{(m_e, n_e)}$$

$$B = \left.\frac{\partial f}{\partial n}\right|_{(m_e, n_e)}$$