Models (continued)

AE353

Spring 2025

Bretl

https://go.aerospace.illinois.edu/ae353-sp25

COURSE
WEBSITE

MODEL OF DYNAMICS

STEP 0 – get EOMs

$$\boxed{J\ddot{q} = \tau}$$

This is the model we are given. We want to rewrite this model in "state-space form" so it looks like this.

$$\dot{x} = Ax + Bu$$

STEP 1 – rewrite EOMs as a set of first-order ODEs

$\ddot{q}$ ← find time derivative of highest order

$v = \dot{q}$ ← define new variables for each time derivative of lower order

$J\dot{v} = \tau$ ← rewrite EOMs in terms of new variables

$\dot{q} = v$ ← add an ODE for each new variable

$$\dot{q} = v$$
$$\dot{v} = (1/J)\tau$$

← collect ODEs together, solving for time derivatives if necessary

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ (1/J)\tau \end{bmatrix}$$

← write in standard form

$$\dot{m} = f(m, n) \leftarrow m = \begin{bmatrix} q \\ v \end{bmatrix} \quad n = [\tau]$$

$$\dot{q} = v$$
$$\dot{v} = (1/J)\tau$$

$$\longleftrightarrow \quad \dot{x} = Ax + Bu$$

STEP 2 - find an equilibrium point

$$0 = v_e$$
$$0 = (1/J)\tau_e$$

← set time derivatives to zero

$v_e = 0 \qquad \tau_e = 0$ ← solve

$q_e = \frac{\pi}{2} \qquad v_e = 0 \qquad \tau_e = 0$ ← pick a solution

$m_e = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix} \qquad n_e = \begin{bmatrix} 0 \end{bmatrix}$ ← write in standard form

STEP 3 - define state and input

$$x = m - m_e = \begin{bmatrix} q \\ v \end{bmatrix} - \begin{bmatrix} q_e \\ v_e \end{bmatrix} = \begin{bmatrix} q - \pi/2 \\ v \end{bmatrix}$$

← $x = m - m_e$

$$u = n - n_e = \begin{bmatrix} \tau \end{bmatrix} - \begin{bmatrix} \tau_e \end{bmatrix} = \begin{bmatrix} \tau \end{bmatrix}$$

← $u = n - n_e$

STEP 4 - compute A and B

We defined the state x and the input u like this:

$$x = \begin{bmatrix} q - (\pi/2) \\ v \end{bmatrix} \qquad u = [\tau]$$

What matrices A and B would make

$$\dot{x} = Ax + Bu \quad \text{and} \quad \begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ (1/J)\,\tau \end{bmatrix}$$

describe the same set of ODEs?

$$\underbrace{\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q - (\pi/2) \\ v \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1/J \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \tau \end{bmatrix}}_{u}$$

$$x = \begin{bmatrix} q - (\pi/2) \\ v \end{bmatrix} \qquad u = \begin{bmatrix} \tau \end{bmatrix}$$

MODEL OF CONTROLLER

$$u = -Kx$$

What matrix K would make

$$u = -Kx \quad \text{and} \quad \tau = -5(q - (\pi/2)) - (1/2)v$$

describe the same controller?

$$\underbrace{\begin{bmatrix} \tau \end{bmatrix}}_{u} = -\underbrace{\begin{bmatrix} 5 & 1/2 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} q - (\pi/2) \\ v \end{bmatrix}}_{x}$$

```
wheel_torque = - 5. * (wheel_angle - (np.pi / 2)) - 0.5 * wheel_velocity
```

STATE-SPACE MODEL (linear and time-invariant)

$$\dot{x} = Ax + Bu$$

state

input

LINEAR STATE FEEDBACK

$$u = -Kx$$

gain matrix