Optimal observers

AE353

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Bretl

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

n_x number of states
n_u number of inputs
n_y number of outputs

OPTIMAL CONTROLLER

$u = -Kx$ where $K = \text{lqr}(A, B, Q_c, R_c)$

diagonal w/ positive numbers

$\left\{ \begin{array}{l} Q_c \text{ is } n_x \times n_x \\ R_c \text{ is } n_u \times n_u \end{array} \right.$

← error
← effort

OPTIMAL OBSERVER

$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$ where $L = \text{lqr}(A^T, C^T, R_o^{-1}, Q_o^{-1})^T$

diagonal w/ positive numbers

$\left\{ \begin{array}{l} Q_o \text{ is } n_y \times n_y \\ R_o \text{ is } n_x \times n_x \end{array} \right.$

← sensors
← dynamics

EXAMPLE - estimate distance of drone to ground

ONE MEASUREMENT

$$y = x \qquad y = 1 \qquad \hat{x} = 1$$

TWO MEASUREMENTS

$n_1 = y_1 - x$

$$y = \begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \qquad \begin{matrix} \leftarrow y_1 = x + n_1 \\ \leftarrow y_2 = x + n_2 \end{matrix} \qquad \begin{matrix} y_1 = 1 \\ y_2 = 2 \end{matrix} \qquad \hat{x} = \cancel{1.5}$$

$$1.1 = 0.9\, y_1 + 0.1\, y_2$$

MEASUREMENT ERROR

$\hat{x}$	n_1	n_2
1.1	-0.1	0.9
1.5	-0.5	0.5
1.8	-0.8	0.2

$y_1 = x + n_1$

$1 = (1.1) + (-0.1)$

THE "COST" OF AN ESTIMATE

$$\underset{x}{\text{minimize}} \; q_1 (n_1)^2 + q_2 (n_2)^2 = \underset{x}{\text{minimize}} \; q_1 (y_1 - x)^2 + q_2 (y_2 - x)^2$$

$$\left.\begin{aligned} y_1 &= x + n_1 \\ y_2 &= x + n_2 \end{aligned}\right\}$$ given y_1 and y_2, choose $\hat{x}$ by solving ...

$$\underset{x}{\text{minimize}} \quad \underbrace{q_1(y_1 - x)^2 + q_2(y_2 - x)^2}_{h(x)}$$

$$\frac{\partial h}{\partial x} = -2q_1(y_1 - x) - 2q_2(y_2 - x) = 0$$

$$= q_1(y_1 - x) + q_2(y_2 - x)$$

$$= q_1 y_1 - q_1 x + q_2 y_2 - q_2 x$$

$$= (q_1 y_1 + q_2 y_2) - (q_1 + q_2)x$$

$$\Rightarrow \quad (q_1 + q_2)x = q_1 y_1 + q_2 y_2 \Rightarrow x = \frac{q_1 y_1 + q_2 y_2}{q_1 + q_2}$$

$$x = \left(\frac{q_1}{q_1 + q_2}\right) y_1 + \left(\frac{q_2}{q_1 + q_2}\right) y_2$$

$$\underset{u[t_0,\infty)}{\text{minimize}} \int_{t_0}^{\infty} \Big(\underbrace{x(t)^T Q_c\, x(t)}_{\text{error}} + \underbrace{u(t)^T R_c\, u(t)}_{\text{effort}} \Big)\, dt$$

$$\boxed{u = -K_c x}$$

subject to $\dot{x}(t) = Ax(t) + Bu(t)$ for $t \in [t_0, \infty)$

$x(t_0) = x_0$

OPTIMAL CONTROLLER

OPTIMAL OBSERVER

$$\underset{x(t_1)}{\text{minimize}} \int_{-\infty}^{t_1} \Big(\underbrace{n(t)^T Q_o\, n(t)}_{\text{sensor noise}} + \underbrace{d(t)^T R_o\, d(t)}_{\text{process disturbance}} \Big)\, dt$$

subject to
$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t) \\ y(t) &= Cx(t) + n(t) \end{aligned}\right\} \text{ for } t \in (-\infty, t_1]$$

SUPPOSE:

$x(t_1) = \alpha$ if $t_1 = t_\alpha$

$x(t_1) = \beta$ if $t_1 = t_\beta$

THEN:

the solution to

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)}, \quad \hat{x}(t_\alpha) = \alpha$$

is $\hat{x}(t_\beta) = \beta$

$$\underset{x(t_1)}{\text{minimize}} \quad \int_{-\infty}^{t_1} \left(n(t)^T Q_0 n(t) + d(t)^T R_0 d(t) \right) dt$$

$$\text{subject to} \quad \left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t) \\ y(t) &= Cx(t) + n(t) \end{aligned}\right\} \text{ for } t \in (-\infty, t_1]$$

↑ IN GENERAL

↓ SPECIFIC EXAMPLE IN CLASS

(scalar system, $A = 0$ $B = 1$ $C = 1$ $D = 0$)

$$\underset{x(t_1)}{\text{minimize}} \quad \int_{-\infty}^{t_1} \left(q\, n(t)^2 + r\, d(t)^2 \right) dt$$

$$\text{subject to} \quad \left.\begin{aligned} \dot{x}(t) &= u(t) + d(t) \\ y(t) &= x(t) + n(t) \end{aligned}\right\} \text{ for } t \in (-\infty, t_1]$$

$$\dot{x} = [0]x + [1]u + d$$

$$y = [1]x + n$$

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$$

$$L = [l]$$

$$L = lqr(A^T, C^T, (R_0)^{-1}, (Q_0)^{-1})^T$$

$$Q = [q] \qquad R = [r]$$

$$\int (n^T Q n + d^T R d)\, dt$$

$$(y - Cx) \qquad (\dot{x} - (Ax + Bu))$$