LQR (details)

AE353

Spring 2023

Bretl

Linear Quadratic Regulator (LQR)

total cost

cost

$$\underset{u_{[t_0,\infty)},\, x_{[t_0,\infty)}}{\boxed{\text{minimize}}} \quad \int_{t_0}^{\infty} \left(x(t)^T Q\, x(t) + u(t)^T R\, u(t) \right) dt$$

subject to

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ x(t_0) &= x_0 \end{aligned}\right\} \text{ constraints}$$

decision variables

WEIGHTS

The minimizer (i.e., the input that achieves minimum cost) is

$$u(t) = -K\,x(t)$$

and the minimum (i.e., the minimum cost) is

$$x_0^T P\, x_0$$

where K and P can be found in python as follows:

```python
def lqr(A, B, Q, R):
    P = linalg.solve_continuous_are(A, B, Q, R)
    K = linalg.inv(R) @  B.T @ P
    return K, P
```

$$\min_{u_{[t_0,\infty)},\, x_{[t_0,\infty)}} \int_{t_0}^{\infty} \left(x(t)^T Q\, x(t) + u(t)^T R\, u(t) \right) dt$$

subject to

$$\dot{x}(t) = A x(t) + B u(t)$$
$$x(t_0) = x_0$$

Why is the cost "quadratic" and what does it really mean?

$x = [x_1]$ $\quad$ $Q = [q_1]$ $\quad$ $u = [u_1]$ $\quad$ $R = [r_1]$

$$x^T Q x + u^T R u = [x_1][q_1][x_1] + [u_1][r_1][u_1]$$
$$= [q_1 x_1^2] + [r_1 u_1^2]$$
$$= [q_1 x_1^2 + r_1 u_1^2]$$

Why is the cost "quadratic" and what does it really mean?

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u = [u_1]$$

$$Q = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix} \qquad R = [r_1]$$

$$x^T Q x + u^T R u = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [u_1][r_1][u_1]$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_1 x_1 + q_3 x_2 \\ q_3 x_1 + q_2 x_2 \end{bmatrix} + [r_1 u_1^2]$$

$$= \left[q_1 x_1^2 + q_3 x_1 x_2 + q_3 x_1 x_2 + q_2 x_2^2 + r_1 u_1^2 \right]$$

$$= \left[q_1 x_1^2 + 2 q_3 x_1 x_2 + q_2 x_2^2 + r_1 u_1^2 \right]$$

What Q and R would produce a given cost?

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u = [u_1]$$

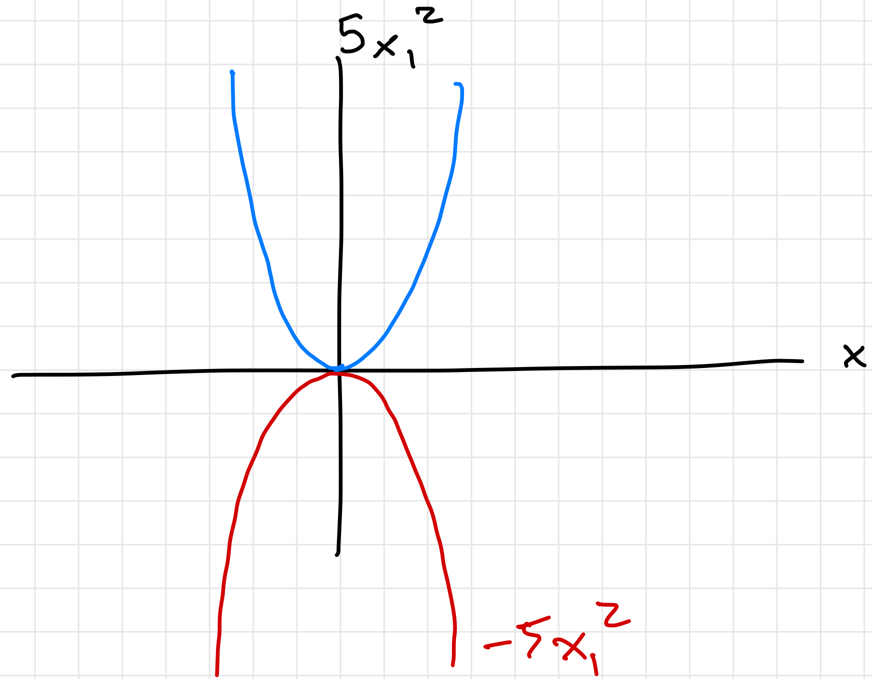
$$Q = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix} \qquad R = [r_1]$$

$$\left[q_1 x_1^2 + 2 q_3 x_1 x_2 + q_2 x_2^2 + r_1 u_1^2 \right]$$

$$x^T Q x + u^T R u = \left[5 x_1^2 - 2 x_1 x_2 + 2 x_2^2 + 1 u_1^2 \right]$$

$$\downarrow$$

$$Q = \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \qquad R = [1]$$



$5x_1^2$
x_1
$-5x_1^2$

Q and R are commonly chosen to be diagonal

all positive

$$Q = \operatorname{diag}(q_1, \ldots, q_{n_x})$$

$$R = \operatorname{diag}(r_1, \ldots, r_{n_u})$$

all positive

Rule of thumb

$$q_i = \left(\frac{1}{x_{i,max}}\right)^2 \qquad r_i = \left(\frac{1}{u_{i,max}}\right)^2$$