Ackermann's method

AE353

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Bretl

$$A = \begin{bmatrix} 2 & 0 & -5 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \qquad K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

The characteristic equation we have is

$$\det(sI - (A - BK)) =$$

$$s^3 + (-2k_2 + k_3 - 7)s^2 + (-5k_1 + 11k_2 - 6k_3 + 15)s + (20k_1 - 9k_2 + 8k_3 - 15).$$

Suppose the characteristic equation we want is

$$s^3 + r_1 s^2 + r_2 s + r_3 .$$

Then we have to solve

$$\begin{aligned} -2k_2 + k_3 - 7 &= r_1 \\ -5k_1 + 11k_2 - 6k_3 + 15 &= r_2 \\ 20k_1 - 9k_2 + 8k_3 - 15 &= r_3 . \end{aligned}$$

$$A = \begin{array}{ccc} -a_1 & -a_2 & -a_3 \\ \multicolumn{3}{c}{\begin{bmatrix} -3 & -2 & -5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \end{array} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

The characteristic equation we have is

$$\det(sI - (A - BK)) = s^3 + \underbrace{(k_1+3)}_{k_1+a_1} s^2 + \underbrace{(k_2+2)}_{k_2+a_2} s + \underbrace{(k_3+5)}_{k_3+a_3}$$

Suppose the characteristic equation we want is

$$s^3 + r_1 s^2 + r_2 s + r_3$$

Then we have to solve

$$\left.\begin{aligned} k_1 + 3 &= r_1 & \qquad k_1 &= r_1 - 3 \\ k_2 + 2 &= r_2 & \qquad k_2 &= r_2 - 2 \\ k_3 + 5 &= r_3 & \qquad k_3 &= r_3 - 5 \end{aligned}\right\} \qquad \begin{aligned} k_1 &= r_1 - a_1 \\ k_2 &= r_2 - a_2 \\ k_3 &= r_3 - a_3 \end{aligned}$$

Controllable Canonical Form (CCF)

$$A = \begin{bmatrix} \begin{bmatrix} -a_1 & \cdots & -a_n \end{bmatrix} \\ \begin{bmatrix} I_{(n-1)\times(n-1)} \end{bmatrix} \begin{bmatrix} 0_{(n\times 1)\times 1} \end{bmatrix} \end{bmatrix} \qquad B = \begin{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \\ \begin{bmatrix} 0_{(n-1)\times 1} \end{bmatrix} \end{bmatrix}$$

Facts

$$\det(sI - A) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$$

$$A - BK = \begin{bmatrix} \begin{bmatrix} -a_1 - k_1 & \cdots & -a_n - k_n \end{bmatrix} \\ \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \end{bmatrix}$$

$$\det(sI - (A - BK)) = s^n + (a_1 + k_1) s^{n-1} + \dots + (a_{n-1} + k_{n-1}) s + (a_n + k_n)$$

Consequence

no symbolic computation!

if you want $\quad s^n + r_1 s^{n-1} + \dots + r_{n-1} s + r_n$

then $\quad k_1 = r_1 - a_1 \quad \dots \quad k_n = r_n - a_n$

If we could put a system in CCF...

$$\dot{x} = Ax + Bu \qquad z = V^{-1}x$$

$$x = Vz$$

$$V\dot{z} = AVz + Bu$$

$$\dot{z} = \underbrace{V^{-1}AV}z + V^{-1}Bu$$

$$\dot{z} = A_{CCF}\,z + B_{CCF}\,u$$

Then...

$$u = -\overbrace{K_{CCF}}^{\text{easy to find}}\,z = -K_{CCF}V^{-1}x$$

$$= -\underbrace{\left(K_{CCF}V^{-1}\right)}_{K \text{ (what we want)}}x$$

How to find A_{CCF}

$$A_{CCF} = \begin{bmatrix} \begin{bmatrix} -a_1 & \cdots & -a_n \end{bmatrix} \\ \begin{bmatrix} I_{(n-1)\times(n-1)} \end{bmatrix} \begin{bmatrix} 0_{(n-1)\times 1} \end{bmatrix} \end{bmatrix} \qquad B_{CCF} = \begin{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \\ \begin{bmatrix} 0_{(n-1)\times 1} \end{bmatrix} \end{bmatrix}$$

$$\det(sI - A_{CCF}) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

$$\det(sI - V^{-1} A V) = \det(sI - A)$$

How to find V given A_{CCF} and B_{CCF}

solve for V^{-1} (that's what we need to find K given K_{CCF})

$$\underbrace{A_{CCF} = V^{-1} A V \qquad B_{CCF} = V^{-1} B}$$

$$B_{CCF} = V^{-1} B$$

$$A_{CCF} B_{CCF} = V^{-1} A \cancel{V V^{-1}} B = V^{-1} A B$$

$$A_{CCF}^2 B_{CCF} = V^{-1} A \cancel{V V^{-1}} A \cancel{V V^{-1}} B = V^{-1} A^2 B$$

$\vdots$

$$A_{CCF}^{n-1} B_{CCF} =$$

$$\underbrace{\begin{bmatrix} B_{CCF} & A_{CCF} B_{CCF} & \cdots & A_{CCF}^{n-1} B_{CCF} \end{bmatrix}}_{W_{CCF}} = V^{-1} \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1} B \end{bmatrix}}_{W}$$

$$\boxed{V^{-1} = W_{CCF} W^{-1}}$$

works as long as W is invertible

ACKERMANN'S METHOD

- Compute the characteristic equation that we want:

$$(s - p_1)\cdots(s - p_n) = s^n + r_1 s^{n-1} + \cdots + r_{n-1}s + r_n$$

- Compute the characteristic equation that we have:

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

- Compute the controllability matrix of the original system (and check that $\det(W) \neq 0$):

$$W = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

- Compute the controllability matrix of the transformed system:

$$W_{\text{ccf}} = \begin{bmatrix} B_{\text{ccf}} & A_{\text{ccf}}B_{\text{ccf}} & \cdots & A_{\text{ccf}}^{n-1}B_{\text{ccf}} \end{bmatrix}$$

  where

$$A_{\text{ccf}} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_{\text{ccf}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Compute the gains for the transformed system:

$$K_{\text{ccf}} = \begin{bmatrix} r_1 - a_1 & \cdots & r_n - a_n \end{bmatrix}$$

- Compute the gains for the original system:

$$K = K_{\text{ccf}} W_{\text{ccf}} W^{-1}$$

The system

$$\dot{x} = Ax + Bu$$

is controllable if

$$W = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

"n" is the number of states (i.e., the size of x)

has full rank.