Diagonalization (Part 2)

AE 353
Spring 2023
Bret1
LAST TIME

\[ \dot{x} = Ax + Bu \quad \text{← model of all dynamics we care about} \]
\[ u = -Kx \quad \text{← model of all controllers we care about} \]

\[ \dot{x} = (A - BK)x \quad \text{← closed-loop system} \]

\[ x(t) = e^{(A - BK)t} x(0) \quad \text{← solution (by matrix exponential)} \]

\[ x(t) \to 0 \text{ as } t \to \infty \quad \text{if and only if all eigenvalues of } A - BK \text{ have negative real part} \]

\[ \uparrow \quad \text{asymptotic stability} \]

\[ \text{OUR GOAL IS TO PROVE THIS} \]
\[ \dot{x} = Fx \quad \text{for which } F \text{ does } x(t) \to 0 \text{ as } t \to \infty \]???

**STRATEGY**

1. Answer this question in the special case when $F$ is diagonal
2. Show how to rewrite (almost) any $F$ as diagonal
3. Answer this question for (almost) any $F$
Suppose \( F \) is diagonal

\[
F = \begin{bmatrix}
s_1 & 0 \\
0 & s_2
\end{bmatrix}
\]

then:

If \( \dot{x} = Fx \) then \( x(t) = e^{Ft}x(0) \)

\[
e^{Ft} = I + Ft + \frac{1}{2!}(Ft)^2 + \ldots
\]

\[
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} + \begin{bmatrix}
s_1 & 0 \\
0 & s_2
\end{bmatrix} + \begin{bmatrix}
\frac{1}{2}(s_1)^2 & 0 \\
0 & \frac{1}{2}(s_2)^2
\end{bmatrix} + \ldots
\]

\[
= \begin{bmatrix}
1 + (s_1)t + \frac{1}{2}(s_1)^2 + \ldots & 0 \\
0 & 1 + s_2t + \frac{1}{2}(s_2)^2 + \ldots
\end{bmatrix}
\]

\[
= \begin{bmatrix}
e^{s_1t} & 0 \\
0 & e^{s_2t}
\end{bmatrix}
\]

when \( F \) is diagonal, \( e^{Ft} \) is easy to find
coordinate invariance

\[ \dot{x} = Fx \]

Plug in \( x = Vz \) for some invertible \( V \) for some invertible \( V \)
old coordinates
new coordinates

\[ V\dot{z} = FVz \]

solve for \( \dot{z} \)

\[ \dot{z} = (V^{-1}FV)z \]

solve for \( z(t) \) with matrix exponential

\[ z(t) = e^{(V^{-1}FV)t} z(0) \]

Plug in \( z = V^{-1}x \) and solve for \( x \)

\[ Vx(t) = e^{(V^{-1}FV)^{-1}t} Vx(0) \]

\[ x(t) = Ve^{(V^{-1}FV)^{-1}t} Vx(0) \]

\[ x(t) = e^{Ft} x(0) \]

These must be equal!

\[ Ve^{(V^{-1}FV)^{-1}t} Vx(0) = e^{Ft} x(0) \]

for any invertible \( V \)
\[
\dot{x} = Fx
\]

for which \( F \) does \( x(t) \to 0 \) as \( t \to \infty \) for any \( x(0) \)?

\[
x(t) = e^{Ft}x(0)
\]

\[ e^{Ft} \to \theta \text{ as } t \to \infty \]

1. Answer this question in the special case when \( F \) is diagonal

\[
F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \quad \Rightarrow \quad e^{Ft} = \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}
\]

\[
s_1 = a + jb
\]

\[
e^{s_1 t} = e^{(a+jb)t} = e^{at}e^{jbt} = e^{at}(\cos(bt) + j \sin(bt))
\]
(2) Show how to rewrite (almost) any $F$ as diagonal

$$x(t) = e^{Ft} x(0)$$

$$x(t) = V e^{(V^{-1} F V)^t} V^{-1} x(0)$$

for any invertible $V$

$$e^{Ft} = V e^{(V^{-1} F V)^t} V^{-1}$$

for any invertible $V$

choose $V$ so $V^{-1} F V$ is diagonal
choose $V$ so $V^{-1}FV$ is diagonal

$$V^{-1}FV = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \Rightarrow FV = V \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

$V = [v_1, v_2]$

$$F \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

$$\begin{bmatrix} Fv_1 \\ Fv_2 \end{bmatrix} = \begin{bmatrix} v_1s_1 \\ v_2s_2 \end{bmatrix} \Rightarrow Fv_1 = v_1s_1, \quad Fv_2 = v_2s_2$$

$v_1$ and $v_2$ are EIGENVECTORS of $F$

$s_1$ and $s_2$ are EIGENVALUES of $F$
for which $F$ does $x(t) \to 0$ as $t \to \infty$ for any $x(0)$?

3. Answer this question for (almost) any $F$

We have shown:

$$x(t) = e^{Ft}x(0) = Ve^{(V^tFV)^t}V^{-1}x(0)$$

This means that

$$e^{Ft} \to \mathbb{0} \quad \text{if and only if} \quad e^{(V^tFV)^t} \to \mathbb{0}$$

for some invertible $V$. If we choose $V$ as a matrix with the eigenvectors of $F$ as its columns (and if all eigenvalues of $F$ are distinct) then

$$e^{(V^tFV)^t} = e^{[s_1 \quad 0]^t} = \begin{bmatrix} e^{s_1t} & 0 \\ 0 & e^{s_2t} \end{bmatrix}$$

where $s_1$ and $s_2$ are the eigenvalues of $F$ (for our $2 \times 2$ example).
The system
\[ \dot{x} = Fx \]
is asymptotically stable if and only if all eigenvalues of \( F \) have negative real part.