Diagonalization (Part 2)

AE353

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Bretl

LAST TIME

$\dot{x} = Ax + Bu$ ← model of all dynamics we care about

$u = -Kx$ ← model of all controllers we care about

$\downarrow$

$\dot{x} = (A - BK)x$ ← closed-loop system

$\downarrow$

$x(t) = e^{(A-BK)t}x(0)$ ← solution (by matrix exponential)

$\vdots$

$\downarrow$

$x(t) \to 0$ as $t \to \infty$ if and only if all eigenvalues of $A-BK$ have negative real part } asymptotic stability

↑

OUR GOAL IS TO PROVE THIS

$\dot{x} = Fx$ ← for which F does $x(t) \to 0$ as $t \to \infty$???

STRATEGY

1. Answer this question in the special case when F is diagonal
2. Show how to rewrite (almost) any F as diagonal
3. Answer this question for (almost) any F

suppose F is diagonal

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

$$\begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix} \begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix}$$

then:

if $\dot{x} = Fx$ then $x(t) = e^{Ft} x(0)$

$$e^{Ft} = I + Ft + \frac{1}{2!}(Ft)^2 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(s_1 t)^2 & 0 \\ 0 & \frac{1}{2}(s_2 t)^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + (s_1 t) + \frac{1}{2}(s_1 t)^2 + \cdots & 0 \\ 0 & 1 + s_2 t + \frac{1}{2}(s_2 t)^2 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}$$

$\leftarrow$ when F is diagonal, e^{Ft} is easy to find

coordinate invariance

solve for $x(t)$ with matrix exponential

$$\dot{x} = Fx$$

$\downarrow$ plug in $x = Vz$ for some invertible V

$$V\dot{z} = FVz$$

$\downarrow$ solve for $\dot{z}$

$$\dot{z} = \left(V^{-1}FV\right)z$$

$\downarrow$ solve for $z(t)$ with matrix exponential

$$z(t) = e^{(V^{-1}FV)t}\, z(0)$$

$\downarrow$ plug in $z = V^{-1}x$ and solve for x

$z(t) = V^{-1}x(t)$

$$V^{-1}x(t) = e^{(V^{-1}FV)t}\, V^{-1}x(0)$$

$$\left.\begin{aligned} x(t) &= Ve^{(V^{-1}FV)t}\, V^{-1}x(0) \\ x(t) &= e^{Ft}\, x(0) \end{aligned}\right\}$$

THESE MUST BE EQUAL!

$$Ve^{(V^{-1}FV)t}\, V^{-1}x(0) = e^{Ft}x(0)$$

for any invertible V

old coordinates — new coordinates

$$\underbrace{\begin{bmatrix} g \\ v \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} g+v \\ g-v \end{bmatrix}}_{z}$$

EXAMPLE OF COORDINATE TRANSFORMATION

$\boxed{\dot{x} = Fx}$ ← for which F does $\underbrace{x(t)}$ → 0 as $t \to \infty$ for $\underline{\text{any}}$ $x(0)$?

$$x(t) = e^{Ft} x(0)$$

$$e^{Ft} \to 0 \text{ as } t \to \infty$$

① Answer this question in the special case when F is diagonal

matrix exp ↓ scalar exp

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \Rightarrow e^{Ft} = \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}$$

$$s_1 = a + jb$$

$$e^{s_1 t} = e^{(a+jb)t} = e^{at} e^{jbt} = \underline{e^{at}}\left(\cos(\underline{bt}) + j\sin(\underline{bt})\right)$$

② Show how to rewrite (almost) any F as diagonal

$$x(t) = e^{Ft}\,x(0)$$

$x = Vz$

$$x(t) = V e^{(V^{-1}FV)t}\,V^{-1}\,x(0)$$

for any invertible V

$$e^{Ft} = V\,e^{(V^{-1}FV)t}\,V^{-1}$$

for any invertible V

choose V so $V^{-1}FV$ is diagonal

choose V so $V^{-1}FV$ is diagonal

$$V^{-1}FV = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \quad \Rightarrow \quad FV = V \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

$$V = [v_1 \quad v_2]$$

$$\underbrace{F[v_1 \quad v_2]}_{} = [v_1 \quad v_2] \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

$$[Fv_1 \quad Fv_2] = [v_1 s_1 \quad v_2 s_2] \longrightarrow \begin{aligned} Fv_1 &= v_1 s_1 \\ Fv_2 &= v_2 s_2 \end{aligned}$$

v_1 and v_2 are EIGENVECTORS of F

s_1 and s_2 are EIGENVALUES of F

$\boxed{\dot{x} = Fx}$ ← for which F does $x(t) \to 0$ as $t \to \infty$ for <u>any</u> $x(0)$?

③ Answer this question for (almost) any F

We have shown:

$$x(t) = e^{Ft} x(0) = V e^{(V^{-1}FV)t} V^{-1} x(0)$$

This means that

$$e^{Ft} \to 0 \quad \text{if and only if} \quad e^{(V^{-1}FV)t} \to 0$$

for some invertible V. If we choose V as a matrix with the eigenvectors of F as its columns (and if all eigenvalues of F are distinct) then

$$e^{(V^{-1}FV)t} = e^{\begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} t} = \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}$$

where s_1 and s_2 are the eigenvalues of F (for our 2×2 example).

The system

$$\dot{x} = Fx$$

is asymptotically stable if and only if all eigenvalues of F have negative real part.

$x(t) \to 0$ as $t \to \infty$ for any $x(0)$