Diagonalization (Part 1)

AE353

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Bretl

LAST TIME

$\dot{x} = Ax + Bu$ ← model of all dynamics we care about

$u = -Kx$ ← model of all controllers we care about

$\downarrow$

$\dot{x} = (A-BK)x$ ← closed-loop system

$\downarrow$

$x(t) = e^{(A-BK)t}x(0)$ ← solution (by matrix exponential)

$\vdots$

$\downarrow$

$x(t) \to 0$ as $t \to \infty$ if and only if all eigenvalues of $A-BK$ have negative real part } asymptotic stability

↑

OUR GOAL IS TO PROVE THIS

$\dot{x} = Fx \leftarrow$ for which F does $x(t) \to 0$ as $t \to \infty$???

STRATEGY

1. Answer this question in the special case when F is diagonal
2. Show how to rewrite (almost) any F as diagonal
3. Answer this question for (almost) any F

suppose F is diagonal

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

$$\begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix}\begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix}$$

then:

if $\dot{x} = Fx$ then $x(t) = e^{Ft} x(0)$

$$e^{Ft} = I + Ft + \frac{1}{2!}(Ft)^2 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(s_1 t)^2 & 0 \\ 0 & \frac{1}{2}(s_2 t)^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + (s_1 t) + \frac{1}{2}(s_1 t)^2 + \cdots & 0 \\ 0 & 1 + s_2 t + \frac{1}{2}(s_2 t)^2 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}$$

$\leftarrow$ when F is diagonal, e^{Ft} is easy to find

coordinate invariance

$$\dot{x} = Fx$$

solve for $x(t)$ with matrix exponential

plug in $x = Vz$ for some invertible V

$$V\dot{z} = FVz$$

solve for $\dot{z}$

$$\dot{z} = \left(V^{-1}FV\right)z$$

solve for $z(t)$ with matrix exponential

$$z(t) = e^{(V^{-1}FV)t}\, z(0)$$

$$z(t) = V^{-1}x(t)$$

plug in $z = V^{-1}x$ and solve for x

$$V^{-1}x(t) = e^{(V^{-1}FV)t}\, V^{-1}x(0)$$

$$x(t) = Ve^{(V^{-1}FV)t}\, V^{-1}x(0)$$

$$x(t) = e^{Ft}\, x(0)$$

THESE MUST BE EQUAL!

$$Ve^{(V^{-1}FV)t}\, V^{-1}x(0) = e^{Ft}\, x(0)$$

for any invertible V

EXAMPLE OF COORDINATE TRANSFORMATION

$$\overbrace{\underbrace{\begin{bmatrix} q \\ v \end{bmatrix}}_{x}}^{\text{old coordinates}} = \underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}_{V} \overbrace{\underbrace{\begin{bmatrix} q+v \\ q-v \end{bmatrix}}_{z}}^{\text{new coordinates}}$$