Matrix exponential and Stability

AE 353
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$$c_1 \ddot{q}_b = \tau - c_2 \sin q_b$$

$$\downarrow$$

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ (\tau - c_2 \sin q_b)/c_1 \end{bmatrix}$$

$$(\tau + c_2(q_b - \pi))/c_1$$

$$\downarrow$$

$$\dot{x} = Ax + Bu \qquad \begin{cases} x = \begin{bmatrix} q_b - \pi \\ v - 0 \end{bmatrix} & u = [\tau - 0] \\ A = \begin{bmatrix} 0 & 1 \\ c_2/c_1 & 0 \end{bmatrix} & B = \begin{bmatrix} 0 \\ 1/c \end{bmatrix} \end{cases}$$

$$\uparrow$$

$$\begin{bmatrix} \dot{q}_b \\ \dot{v} \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ c_2/c_1 & 0 \end{bmatrix} \begin{bmatrix} q_b - \pi \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1/c_1 \end{bmatrix} [\tau] = \begin{bmatrix} v \\ (c_2/c_1)(q_b - \pi) + (1/c_1)\tau \end{bmatrix}$$

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ c_2/c_1 & 0 \end{bmatrix} \begin{bmatrix} q - \pi \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1/c_1 \end{bmatrix} \begin{bmatrix} \tau \end{bmatrix} = \begin{bmatrix} v \\ (c_2/c_1)(q-\pi) + (1/c_1)\tau \end{bmatrix}$$

$$\begin{bmatrix} \dot{v} \\ \dot{q} \end{bmatrix} \approx \underline{\begin{bmatrix} 0 & c_2/c_1 \\ 1 & 0 \end{bmatrix}} \underline{\begin{bmatrix} v \\ q - \pi \end{bmatrix}} + \underline{\begin{bmatrix} 1/c_1 \\ 0 \end{bmatrix}} \underline{\begin{bmatrix} \tau \end{bmatrix}} = \underline{\begin{bmatrix} (c_2/c_1)(q-\pi) + (1/c_1)\tau \\ v \end{bmatrix}}$$

CAN WE PREDICT WHAT WILL HAPPEN WITHOUT SIMULATION?

open-loop system $\dot{x} = Ax + Bu$ ← state space model

controller $u = -Kx$ ← linear state feedback

closed-loop system
$$\underbrace{\begin{aligned}\dot{x} &= Ax + B(-Kx)\\ &= (A - BK)x\end{aligned}}$$

$$\dot{x} = (A - BK)\, x$$

CAN WE PREDICT WHAT WILL HAPPEN WITHOUT SIMULATION?

$$\boxed{\dot{x} = (a-bk)x} \leftarrow x(t) = e^{(a-bk)t}\,x(0)$$

SCALAR exp. func.

$$\dot{x} = 3x$$

$$x(t) = e^{3t}x(0) \leftarrow \dot{x} = 3\underbrace{e^{3t}x(0)}_{x}$$

$$e^{m} = 1 + m + \frac{1}{2}m^2 + \frac{1}{6}m^3 + \dots = \sum_{k=0}^{\infty}\frac{m^k}{k!}$$

$$\underbrace{\dot{x} = (A-BK)x}_{\dot{x} = Fx} \leftarrow x(t) = e^{(A-BK)t}\,x(0)$$

MATRIX exp. func.

$$e^{M} = I + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + \dots = \sum_{k=0}^{\infty}\frac{M^k}{k!}$$

CAN WE PREDICT WHAT WILL HAPPEN WITHOUT FINDING $x(t)$?

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The closed-loop system

$$\dot{x} = (A - BK)\,x$$

is asymptotically stable if and only if all eigenvalues of

$$A - BK$$

have negative real part.