Day 26

Optimal observers

AE353

Spring 2022

Bretl

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

← n_x number of states
n_u number of inputs
n_y number of outputs

OPTIMAL CONTROLLER

$$u = -Kx \qquad \text{where} \qquad K = lqr(A, B, Q_c, R_c)$$

diagonal w/ positive numbers

- Q_c is $n_x \times n_x$ ← error
- R_c is $n_u \times n_u$ ← effort

OPTIMAL OBSERVER

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y) \qquad \text{where} \qquad L = lqr(A^T, C^T, R_o^{-1}, Q_o^{-1})^T$$

diagonal w/ positive numbers

- Q_o is $n_y \times n_y$ ← sensors
- R_o is $n_x \times n_x$ ← dynamics

height of quadrotor

$y = x$ $\quad y = 1$

$y_1 = x + n_1 \quad y_1 = 1$

$y_2 = x + n_2 \quad y_2 = 2$

$\hat{x} = 1$

$\hat{x} =$ ~~1.5~~ ~~2.0~~ 1.8 ? 1.9 ?

$\hat{x}$	n_1	n_2
1.5	−0.5	0.5
2.0	−1.0	0.0
1.8	−0.8	0.2

$\left(\frac{1}{\sigma_1}\right)^2 \quad \left(\frac{1}{\sigma_2}\right)^2$

minimize $\quad q_1(y_1 - x)^2 + q_2(y_2 - x)^2$
x

$f(x)$

minimize $\quad q_1 n_1^2 + q_2 n_2^2$
x

subject to $\quad y_1 = x + n_1$
$y_2 = x + n_2$

$$\frac{\partial f}{\partial x} = -2q_1(y_1 - x) - 2q_2(y_2 - x)$$
$$= -2\left((q_1 y_1 + q_2 y_2) - x(q_1 + q_2)\right)$$
$$= 0$$

$$x = \frac{q_1 y_1 + q_2 y_2}{q_1 + q_2}$$
$$= \left(\frac{q_1}{q_1 + q_2}\right) y_1 + \left(\frac{q_2}{q_1 + q_2}\right) y_2$$

minimize $u_{[t_0,\infty)}$

$$\int_{t_0}^{\infty} \left(\underbrace{x(t)^T Q_c\, x(t)}_{\text{error}} + \underbrace{u(t)^T R_c\, u(t)}_{\text{effort}} \right) dt$$

subject to $\dot{x}(t) = Ax(t) + Bu(t)$ for $t \in [t_0, \infty)$

$x(t_0) = x_0$

$u = -Kx$

OPTIMAL CONTROLLER

OPTIMAL OBSERVER

minimize $x(t_1), n_{(-\infty,t_1]}, d_{(-\infty,t_1]}$

$$\int_{-\infty}^{t_1} \left(\underbrace{n(t)^T Q_o\, n(t)}_{\text{sensor noise}} + \underbrace{d(t)^T R_o\, d(t)}_{\text{process disturbance}} \right) dt$$

subject to

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t) \\ y(t) &= Cx(t) + n(t) \end{aligned}\right\} \text{ for } t \in (-\infty, t_1]$$

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$$

$$d(t) = \dot{x}(t) - (Ax(t) + Bu(t))$$

$$-n(t) = Cx(t) - y(t)$$

integrate this starting at $\hat{x}(t_\alpha) = \alpha$ then the solution is $\hat{x}(t_\beta) = \beta$

if $t_1 = t_\alpha$ then $x(t_1) = \alpha$

if $t_1 = t_\beta$ then $x(t_1) = \beta$

$$\underset{x(t_1),\, n_{(-\infty,t_1]},\, d_{(-\infty,t_1]}}{\text{minimize}} \quad \int_{-\infty}^{t_1} \left(n(t)^T Q_0 n(t) + d(t)^T R_0 d(t) \right) dt$$

$$\text{subject to} \quad \left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t) \\ y(t) &= Cx(t) + n(t) \end{aligned}\right\} \text{ for } t \in (-\infty, t_1]$$

↑ IN GENERAL

↓ SPECIFIC EXAMPLE IN CLASS (scalar system, $A = 0$ $B = 1$ $C = 1$ $D = 0$)

$$\underset{x(t_1),\, n_{(-\infty,t_1]},\, d_{(-\infty,t_1]}}{\text{minimize}} \quad \int_{-\infty}^{t_1} \left(q\, n(t)^2 + r\, d(t)^2 \right) dt$$

$$\text{subject to} \quad \left.\begin{aligned} \dot{x}(t) &= u(t) + d(t) \\ y(t) &= x(t) + n(t) \end{aligned}\right\} \text{ for } t \in (-\infty, t_1]$$