Day 17

Yet more LQR

AE353

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$$\underset{u_{[t_0,\infty)}}{\text{minimize}} \quad \overbrace{\int_{t_0}^{\infty} \left(q\,x(t)^2 + r\,u(t)^2 \right) dt}^{\text{total cost}}$$

cost

decision variable

$$\text{subject to} \quad \left.\begin{aligned} \dot{x} &= ax + bu \\ x(t_0) &= x_0 \end{aligned}\right\} \text{constraints}$$

the solution is linear state feedback

$$u(t) = -k\,x(t)$$

for a particular choice of k that can be found with two lines of python code

Linear Quadratic Regulator (LQR)

$$\underset{u[t_0,\infty)}{\text{minimize}} \quad \int_{t_0}^{\infty} \Big(\overset{1\times n_x}{x(t)^T} Q \overset{n_x \times 1}{x(t)} + u(t)^T R\, u(t) \Big)\, dt$$

$$\text{subject to} \quad \dot{x}(t) = A x(t) + B u(t)$$

$$x(t_0) = x_0$$

The minimizer (i.e., the input that achieves minimum cost) is

$$u(t) = -K\, x(t)$$

and the minimum (i.e., the minimum cost) is

$$x_0^T P x_0$$

where K and P can be found in python as follows:

```
def lqr(A, B, Q, R):
    P = linalg.solve_continuous_are(A, B, Q, R)
    K = linalg.inv(R) @  B.T @ P
    return K, P
```

Why is the cost "quadratic" and what does it really mean?

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad u = [u_1]$

$Q = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix}$ $\quad R = [r_1]$

$$x^TQx + u^TRu = [x_1 \quad x_2]\begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [u_1][r_1][u_1]$$

$$= [x_1 \quad x_2]\begin{bmatrix} q_1x_1 + q_3x_2 \\ q_3x_1 + q_2x_2 \end{bmatrix} + r_1u_1^2$$

$$= q_1x_1^2 + q_3x_1x_2 + q_3x_1x_2 + q_2x_2^2 + r_1u_1^2$$

$$= q_1x_1^2 + 2\cancel{q_3x_1x_2} + q_2x_2^2 + r_1u_1^2$$

$5x_1^2 - 2x_1x_2 + 3x_2^2 + u_1^2$

What Q and R would produce a given cost?

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u = [u_1]$$

$$Q = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix} \qquad R = [r_1]$$

$$x^T Q x + u^T R u = 5x_1^2 - 2x_1x_2 + 2x_2^2 + 1u_1^2$$

$$\downarrow$$

$$Q = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \qquad R = [\quad]$$

Q and R are commonly chosen to be diagonal

all positive

$$Q = \mathrm{diag}(q_1, \ldots, q_{n_x})$$

$$R = \mathrm{diag}(r_1, \ldots, r_{n_u})$$

all positive