Day 17
Yet more LQR

AE353
Spring 2022
Bret1
Minimize \( u_{\{t_0, \infty\}} \)

Subject to

\[
\begin{align*}
\dot{x} &= ax + bu \\
x(t_0) &= x_0
\end{align*}
\]

Cost

\[
\int_{t_0}^{\infty} (q_x x(t)^2 + r_u u(t)^2) \, dt
\]

Total cost

The solution is linear state feedback

\[ u(t) = -k \cdot x(t) \]

For a particular choice of \( k \) that can be found with two lines of python code.
Linear Quadratic Regulator (LQR)

\[
\begin{align*}
\text{minimize} & \quad \int_{t_0}^{\infty} \left( x(t)^T Q x(t) + u(t)^T R u(t) \right) \, dt \\
\text{subject to} & \quad \dot{x}(t) = A x(t) + B u(t) \\
& \quad x(t_0) = x_0
\end{align*}
\]

The minimizer (i.e., the input that achieves minimum cost) is
\[
u(t) = -K x(t)
\]
and the minimum (i.e., the minimum cost) is
\[
x_0^T P x_0
\]
where \( K \) and \( P \) can be found in Python as follows:

```python
def lqr(A, B, Q, R):
    P = linalg.solve_continuous_are(A, B, Q, R)
    K = linalg.inv(R) @ B.T @ P
    return K, P
```
Why is the cost "quadratic" and what does it really mean?

\[
x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u = [u_1]
\]

\[
Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \quad R = [r_1]
\]

\[
x^T Q x + u^T R u = [x_1 \ x_2] \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [u_1][r_1][u_1]
\]

\[
= x_1^2 q_{11} + 2x_1 x_2 q_{12} + x_2^2 q_{22} + r_1 u_1^2
\]

\[
= 5x_1^2 - 2x_1 x_2 + 3x_2^2 + u_1^2
\]
What $Q$ and $R$ would produce a given cost?

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ \quad \quad $u = \begin{bmatrix} u_1 \end{bmatrix}$

$Q = \begin{bmatrix} q_1 & q_3 \\ q_3 & q_2 \end{bmatrix}$ \quad \quad $R = \begin{bmatrix} r_1 \end{bmatrix}$

$x^TQx + u^TRu = 5x_1^2 - 2x_1x_2 + 2x_2^2 + u_1^2$

$Q = \begin{bmatrix} \end{bmatrix}$ \quad \quad $R = \begin{bmatrix} \end{bmatrix}$
Q and R are commonly chosen to be diagonal

\[ Q = \text{diag} (q_1, \ldots, q_n) \]

\[ R = \text{diag} (r_1, \ldots, r_n) \]

all positive