Day 13

Ackermann's method

AE353

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Bretl

The eigenvalues of a matrix are the roots of its characteristic polynomial

$$\left.\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ u &= -\begin{bmatrix} 20 & 9 \end{bmatrix} x \end{aligned}\right\}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 20 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}$$

$$\det(sI - (A - BK))$$

$$= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} s & -1 \\ 20 & s+9 \end{bmatrix} \right)$$

$$= s^2 + 9s + 20 = (s+4)(s+5) \leftarrow \quad s_1 = -4 \quad s_2 = -5$$

One way to place eigenvalues is to equate coefficients of the characteristic polynomial

$K = \begin{bmatrix} 6 & 5 \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

← find k_1 and k_2 to put closed-loop eigenvalues at $\boxed{-2}$ and $\boxed{-3}$

$$u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} x$$

What do we want?

$$(s-(-2))(s-(-3)) = (s+2)(s+3) = \boxed{s^2 + 5s + 6}$$

What do we have?

$k_2 = 5$

$$A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

$k_1 = 6$

$$\det(sI - (A-BK)) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} s & -1 \\ k_1 & s+k_2 \end{bmatrix}\right) = \boxed{s^2 + k_2 s + k_1}$$

$$(s - (-5 + 3j))(s - (-5 - 3j))$$

It is possible to automate the process of eigenvalue placement

See python demo

Controllable Canonical Form (CCF)

$$K = [k_1 \ k_2 \ \cdots \ k_n]$$

$$A = \begin{bmatrix} [-a_1 & \cdots & -a_n] \\ [\ I_{(n-1)\times(n-1)}\ ] & [\ 0_{(n-1)\times 1}\ ] \end{bmatrix} \qquad B = \begin{bmatrix} [\ 1\ ] \\ [\ 0_{(n-1)\times 1}\ ] \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 & 10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Facts

$$\det(sI - A) = \boxed{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$A - BK = \begin{bmatrix} [-a_1 - k_1 & \cdots & -a_n - k_n] \\ [\ I\ ] & [\ 0\ ] \end{bmatrix}$$

$$\det(sI - (A - BK)) = \boxed{s^n + (a_1 + k_1)s^{n-1} + \cdots + (a_{n-1} + k_{n-1})s + (a_n + k_n)}$$

Consequence

if you want $\quad s^n + r_1 s^{n-1} + \cdots + r_{n-1}s + r_n$

no symbolic computation!

then $\quad k_1 = r_1 - a_1 \quad \cdots \quad k_n = r_n - a_n$

If we could put a system in CCF...

$$\dot{x} = Ax + Bu$$

$$\boxed{x = Vz} \longleftarrow z = V^{-1}x$$

$$\boxed{\dot{z} = A_{CCF}\, z + B_{CCF}\, u}$$

Then ...

$$u = -\overbrace{K_{CCF}}^{\text{easy to find}}\, z$$

$$= -\underbrace{\left(K_{CCF} V^{-1}\right)}_{K \text{ (what we want)}} x$$

How?

solve for V^{-1} (that's what we need to find K given K_{CCF})

$$\overbrace{A_{CCF} = V^{-1}AV \qquad\qquad B_{CCF} = V^{-1}B}$$

$$\dot{x} = Ax + Bu$$

$$x = Vz$$

$$\downarrow$$

$$\dot{z} = \underbrace{V^{-1}AV}_{A_{CCF}}\, z + \underbrace{V^{-1}B}_{B_{CCF}}\, u$$

$$B_{CCF} = V^{-1}B$$

$$A_{CCF}B_{CCF} = (V^{-1}A\underbrace{V)\,V^{-1}}B = V^{-1}AB$$

$$A_{CCF}^{2}B_{CCF} = V^{-1}A\cancel{V\,V^{-1}}A\cancel{V\,V^{-1}}B = V^{-1}A^{2}B$$

$$A_{CCF}^{n-1}B_{CCF} = V^{-1}A^{n-1}B$$

$$\underbrace{\begin{bmatrix} B_{CCF} & A_{CCF}B_{CCF} & \cdots & A_{CCF}^{n-1}B_{CCF} \end{bmatrix}}_{W_{CCF}} = V^{-1}\underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{W}$$

$$\boxed{V^{-1} = W_{CCF}\,W^{-1}}$$

works as long as is invertible

ACKERMANN'S METHOD

- Compute the characteristic equation that we want:

$$(s-p_1)\cdots(s-p_n) = s^n + r_1 s^{n-1} + \cdots + r_{n-1}s + r_n$$

- Compute the characteristic equation that we have:

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

- Compute the controllability matrix of the original system (and check that $\det(W) \neq 0$):

$$W = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

- Compute the controllability matrix of the transformed system:

$$W_{\text{ccf}} = \begin{bmatrix} B_{\text{ccf}} & A_{\text{ccf}}B_{\text{ccf}} & \cdots & A_{\text{ccf}}^{n-1}B_{\text{ccf}} \end{bmatrix}$$

  where

$$A_{\text{ccf}} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_{\text{ccf}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Compute the gains for the transformed system:

$$K_{\text{ccf}} = \begin{bmatrix} r_1 - a_1 & \cdots & r_n - a_n \end{bmatrix}$$

- Compute the gains for the original system:

$$K = K_{\text{ccf}} W_{\text{ccf}} W^{-1}$$