Day 8

AE 353
Spring 2022
Bret1
LAST TIME

\[ x = Ax + Bu \] ← model of all dynamics we care about
\[ u = -Kx \] ← model of all controllers we care about

\[ \dot{x} = (A - BK)x \] ← closed-loop system

\[ x(t) = e^{(A-BK)t} x(0) \] ← solution (by matrix exponential)

\[ x(t) \to 0 \text{ as } t \to \infty \] if and only if all eigenvalues of \( A - BK \) have negative real part

\} \text{ asymptotic stability}
ACTIVITY - PRACTICE DERIVATION
\[ \dot{x} = Fx \]

for which \( F \) does \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty \)???

\[ F = (A - BK) \]

**STRATEGY**

1. Answer this question in the special case when \( F \) is diagonal

2. Show how to rewrite (almost) any \( F \) as diagonal

3. Answer this question for (almost) any \( F \)
Suppose $F$ is diagonal.

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

Then:

$$e^{Ft} = I + Ft + \frac{1}{2}(Ft)^2 + \ldots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} s_1t & 0 \\ 0 & s_2t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2s_1^2t^2 & 0 \\ 0 & 2s_2^2t^2 \end{bmatrix} + \ldots$$

$$= \begin{bmatrix} 1+s_1t + \frac{1}{2}(s_1t)^2 + \ldots & 0 \\ 0 & 1+s_2t + \frac{1}{2}(s_2t)^2 + \ldots \end{bmatrix}$$

$$= \begin{bmatrix} e^{s_1t} & 0 \\ 0 & e^{s_2t} \end{bmatrix}$$

When $F$ is diagonal, $e^{Ft}$ is easy to find.
coordinate invariance

\[
\begin{align*}
\dot{x} &= Fx \\
V\dot{z} &= FVz \\
\end{align*}
\]

plug in \( x = Vz \) for some invertible \( V \)

\[
\begin{align*}
\dot{z} &= V^{-1}FVz \\
\end{align*}
\]
solve for \( z \)

\[
\begin{align*}
z(t) &= e^{(V^{-1}FV)^t}z(0) \\
\end{align*}
\]

plug in \( z = V^{-1}x \) and solve for \( x \)

\[
\begin{align*}
V^{-1}x(t) &= e^{(V^{-1}FV)^t}V^{-1}x(0) \\
\end{align*}
\]

\[
\begin{align*}
x(t) &= \left[ V e^{(V^{-1}FV)^t} V^{-1} \right] x(0) \\
\end{align*}
\]

\[
\begin{align*}
x(t) &= e^{Ft} x(0) \\
\end{align*}
\]

\[
\begin{align*}
e^{Ft} &= Ve^{(V^{-1}FV)^t} V \\
\end{align*}
\]
who cares?

\[ eF^* = V e^{(V^*FV)^*} V^{-1} \]

this is easy to find if \( V^*FV \) is diagonal
so let's choose \( V \) so this is true

our goal - find invertible \( V \) such that

\[ V^{-1}FV = \text{diag} (s_1, \ldots, s_n) \]

← this is the same as

\[ FV = V \text{diag} (s_1, \ldots, s_n) \]

for example, suppose \( F \) is \( 2 \times 2 \):

\[ V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \quad \text{diag} (s_1, s_2) = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \]

← eigenvalues + eigenvectors

\[ F V = \begin{bmatrix} Fv_1 & Fv_2 \end{bmatrix} \quad V \text{diag} (s_1, s_2) = \begin{bmatrix} v_1 s_1 & v_2 s_2 \end{bmatrix} \]

\[ L \text{ columns of } V \text{ (both are } 2 \times 1) \]
if the eigenvalues $s_1, \ldots, s_n$ of $F$ are all distinct and we define a matrix

$$V = [v_1, \ldots, v_n]$$

with the corresponding eigenvectors in each column then

$$x(t) = e^{F t} x(0) = V e^{\text{diag}(s_1, \ldots, s_n) t} V^{-1} x(0)$$

$$= V e^{\begin{pmatrix} s_1 t & \cdots & s_n t \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n t \end{pmatrix}} V^{-1} x(0)$$

what if $s = a + j b$ ???

$$e^{(a+jb)t} = e^{at} e^{jbt}$$

$$= e^{at} (\cos (bt) + j \sin (bt))$$
The system

\[ \dot{x} = Fx \]

is asymptotically stable if and only if all of \( F \) have

\[ x(t) \to 0 \quad \text{as} \quad t \to \infty \]
ACTIVITY - WISH LIST