Day 8

AE353

Spring 2022

Bretl

LAST TIME

$\dot{x} = Ax + Bu$ ← model of all dynamics we care about

$u = -Kx$ ← model of all controllers we care about

$\downarrow$

$\dot{x} = (A-BK)x$ ← closed-loop system

$\downarrow$

$x(t) = e^{(A-BK)t}x(0)$ ← solution (by matrix exponential)

$\vdots$

$\downarrow$

$x(t) \to 0$ as $t \to \infty$ if and only if all eigenvalues of $A-BK$ have negative real part } asymptotic stability

THIS TIME

ACTIVITY – PRACTICE DERIVATION

$$\boxed{\dot{x} = Fx}$$ ← for which F does $x(t) \to 0$ as $t \to \infty$???

$F = (A - BK)$

STRATEGY

1. Answer this question in the special case when F is diagonal
2. Show how to rewrite (almost) any F as diagonal
3. Answer this question for (almost) any F

suppose F is diagonal

NOT DIAGONAL

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \qquad \cancel{\begin{bmatrix} 0 & s_2 \\ s_1 & 0 \end{bmatrix}}$$

$\dot{x} = Fx$

$x(t) = e^{Ft} x(0)$

$\frac{1}{2}(FF)t^2$

then:

$$e^{Ft} = I + Ft + \frac{1}{2!}(Ft)^2 + \ldots$$

matrix exponential

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2} s_1^2 t^2 & 0 \\ 0 & \frac{1}{2} s_2^2 t^2 \end{bmatrix} + \ldots$$

$(s_1 t)^2$

$(s_2 t)^2$

$$= \begin{bmatrix} 1 + s_1 t + \frac{1}{2}(s_1 t)^2 + \ldots & 0 \\ 0 & 1 + s_2 t + \frac{1}{2}(s_2 t)^2 + \ldots \end{bmatrix}$$

$$= \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}$$

scalar exponential

$\leftarrow$ when F is diagonal, e^{Ft} is easy to find

coordinate invariance

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{"}V\text{"}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

solve for $x(t)$ with matrix exponential

$$\dot{x} = Fx$$

$\downarrow$ plug in $\boxed{x = Vz}$ for some invertible V

$$V\dot{z} = FVz$$

$\downarrow$ solve for $\dot{z}$

$$\dot{z} = V^{-1}FVz$$

$\downarrow$ solve for $z(t)$ with matrix exponential

$$z(t) = e^{(V^{-1}FV)t} z(0)$$

$\downarrow$ plug in $z = V^{-1}x$ and solve for x

$$V^{-1}x(t) = e^{(V^{-1}FV)t} V^{-1}x(0)$$

$$x(t) = \boxed{V e^{(V^{-1}FV)t} V^{-1}} x(0)$$

$$x(t) = \boxed{e^{Ft}} x(0)$$

$$e^{Ft} = V e^{(V^{-1}FV)t} V^{-1}$$

who cares?

$$e^{Ft} = V\underbrace{e^{(V^{-1}FV)t}}\,V^{-1}$$

this is easy to find if $V^{-1}FV$ is diagonal
so let's choose V so this is true

our goal – find invertible V such that

$$V^{-1}FV = \text{diag}(s_1, \dots, s_n)$$

← this is the same as
$FV = V\,\text{diag}(s_1, \dots, s_n)$

for example, suppose F is 2×2:

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \qquad \text{diag}(s_1, s_2) = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

columns of V (both are 2×1)

then: $\quad Fv_1 = v_1 s_1 \qquad Fv_2 = v_2 s_2$ ← eigenvalues + eigenvectors

$$FV = \begin{bmatrix} Fv_1 & Fv_2 \end{bmatrix} \qquad V\,\text{diag}(s_1, s_2) = \begin{bmatrix} v_1 s_1 & v_2 s_2 \end{bmatrix}$$

if

the eigenvalues $s_1, \dots, s_n$ of F are all <u>distinct</u>
and we define a matrix

$$V = [v_1 \ \cdots \ v_n]$$

with the corresponding eigenvectors in each column

then

$$x(t) = e^{Ft} x(0) = V \underbrace{e^{\mathrm{diag}(s_1,\dots,s_n)\,t}}_{\mathrm{diag}\left(e^{s_1 t}, \dots, e^{s_n t}\right)} V^{-1} x(0)$$

what if $s = a + jb$???

$$e^{(a+jb)t} = e^{at} e^{jbt}$$

$$= e^{at} \left(\cos(bt) + j \sin(bt) \right)$$

The system

$$\dot{x} = Fx$$

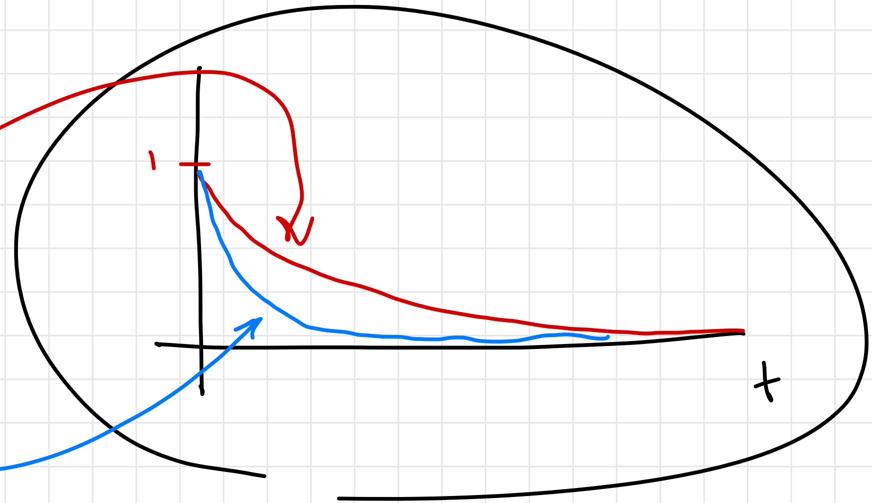
is asymptotically stable if and only if all of F have

$x(t) \to 0$ as $t \to \infty$

ACTIVITY - WISH LIST

$e^{s_1 t}$

e^{-5t}



e^{-50t}