AE353 (Spring 2021)

Day 18

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Acker

$\downarrow$

Controllability

Controllable Canonical Form (CCF)

$$A = \begin{bmatrix} \begin{bmatrix} -a_1 & \cdots & -a_n \end{bmatrix} \\ \begin{bmatrix} I_{(n-1)\times(n-1)} \end{bmatrix} \begin{bmatrix} 0_{(n-1)\times 1} \end{bmatrix} \end{bmatrix} \qquad B = \begin{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \\ \begin{bmatrix} 0_{(n-1)\times 1} \end{bmatrix} \end{bmatrix}$$

Facts

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

$$A - BK = \begin{bmatrix} \begin{bmatrix} -a_1 - k_1 & \cdots & -a_n - k_n \end{bmatrix} \\ \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \end{bmatrix}$$

$$\det(sI - (A - BK)) = s^n + (a_1 + k_1)s^{n-1} + \cdots + (a_{n-1} + k_{n-1})s + (a_n + k_n)$$

Consequence

if you want $s^n + r_1 s^{n-1} + \cdots + r_{n-1}s + r_n$

then $k_1 = r_1 - a_1 \quad \cdots \quad k_n = r_n - a_n$

no symbolic computation!

If we could put a system in CCF...

$$\dot{x} = Ax + Bu$$

$$\boxed{x = Vz} \leftarrow z = V^{-1}x$$

$$\dot{z} = A_{CCF}\, z + B_{CCF}\, u$$

$$\dot{x} = Ax + Bu$$

$$V\dot{z} = AVz + Bu$$

$$\dot{z} = \underbrace{\boxed{V^{-1}AV}}_{A_{CCF}} z + \underbrace{\boxed{V^{-1}B}}_{B_{CCF}} u$$

Then...

$$u = -\overbrace{K_{CCF}}^{\text{easy to find}}\, z$$

$$= -\underbrace{K_{CCF}\, V^{-1}}_{K \text{ (what we want)}}\, x$$

$$A_{CCF} - B_{CCF} K_{CCF}$$

$$A = \begin{bmatrix} -1 & 2 \\ 5 & -10 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ 1 & 0 \end{array}$$

How?

$$\underbrace{A_{CCF} = V^{-1}AV \qquad B_{CCF} = V^{-1}B}_{\text{solve for } V^{-1} \text{ (that's what we need to find } K \text{ given } K_{CCF})}$$

$$\rightarrow \quad B_{CCF} = V^{-1}B$$

$$\rightarrow \quad A_{CCF}B_{CCF} = (V^{-1}AV)(V^{-1}B) = V^{-1}A\cancel{VV^{-1}}B = V^{-1}AB$$

$$\rightarrow \quad A_{CCF}^2B_{CCF} = V^{-1}A\cancel{VV^{-1}}A\cancel{VV^{-1}}B = V^{-1}A^2B$$

$$\vdots$$

$$\rightarrow \quad A_{CCF}^{n-1}B_{CCF} = V^{-1}A^{n-1}B$$

$$\underbrace{\begin{bmatrix} B_{CCF} & A_{CCF}B_{CCF} & A_{CCF}^2B_{CCF} & \cdots & A_{CCF}^{n-1}B_{CCF} \end{bmatrix}}_{W_{CCF} \leftarrow} = V^{-1}\underbrace{\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}}_{W \leftarrow}$$

$$\boxed{V^{-1} = W_{CCF}W^{-1}}$$

works as long as W is invertible

ACKERMANN'S METHOD

- Compute the characteristic equation that we want:

$$(s-p_1)\cdots(s-p_n) = s^n + r_1 s^{n-1} + \cdots + r_{n-1}s + r_n$$

np. poly

- Compute the characteristic equation that we have:

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

np. poly

- Compute the controllability matrix of the original system (and check that $\det(W) \neq 0$):

$$W = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

← np. block

- Compute the controllability matrix of the transformed system:

$$W_{\text{ccf}} = \begin{bmatrix} B_{\text{ccf}} & A_{\text{ccf}}B_{\text{ccf}} & \cdots & A_{\text{ccf}}^{n-1}B_{\text{ccf}} \end{bmatrix}$$

← np. block

where

$$A_{\text{ccf}} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_{\text{ccf}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Compute the gains for the transformed system:

$$K_{\text{ccf}} = \begin{bmatrix} r_1 - a_1 & \cdots & r_n - a_n \end{bmatrix}$$

- Compute the gains for the original system:

$$K = K_{\text{ccf}} W_{\text{ccf}} W^{-1}$$

The system

$$\dot{x} = Ax + Bu$$

is controllable if

$$W = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

"n" is the number of states (i.e., the size of x)

has full rank.

in python this means "np.linalg.matrix_rank(W)" is the same as "n"

if there is only one input, W is square, and so "full rank" and "invertible" mean the same thing — so, for a system with only one input, you can simply check if $\det(W) \neq 0$