

AE353 (Spring 2021)

Day 11 - Stability

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nonlinear ODEs



put in standard form

$$\dot{m} = f(m, n)$$

$$l = g(m, n)$$



linearize

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

$$x = m - m_e$$

$$u = n - n_e$$

$$y = l - g(m_e, n_e)$$

$$0 = f(m_e, n_e)$$



choose input

$$u = -Kx$$



find closed-loop system

$$\dot{x} = (A - BK)x = Fx$$

$$y = (C - DK)x = Gx$$



solve by matrix exponential

$$\begin{aligned}x(t) &= e^{F(t-t_0)} x(t_0) \\ y(t) &= C e^{F(t-t_0)} x(t_0)\end{aligned}$$



for which choices of  $K$  does  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  ???

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## STRATEGY

- ① Answer this question in the special case when  $F$  is diagonal
- ② Show how to rewrite (almost) any  $F$  as diagonal
- ③ Answer this question for (almost) any  $F$

$$\dot{x} = Fx$$

suppose  $F$  is diagonal

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

then:

$$e^{Ft} = I + Ft + \frac{1}{2!}(Ft)^2 + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(s_1 t)^2 & 0 \\ 0 & \frac{1}{2}(s_2 t)^2 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 + s_1 t + \frac{1}{2}(s_1 t)^2 + \dots & 0 \\ 0 & 1 + s_2 t + (s_2 t)^2 + \dots \end{bmatrix}$$

$$= \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}$$

← when  $F$  is diagonal,  
 $e^{Ft}$  is easy to find

# coordinate invariance

$$z = V^{-1}x$$

plug in  $x = Vz$  for some invertible  $V$

$$V\dot{z} = FVz$$

solve for  $\dot{z}$

$$\dot{z} = V^{-1}FVz$$

solve for  $z(t)$  with matrix exponential

$$z(t) = e^{V^{-1}FVt} z(0)$$

plug in  $z = V^{-1}x$  and solve for  $x$

$$V^{-1}x(t) = e^{V^{-1}FVt} V^{-1}x(0)$$

$$x(t) = V e^{V^{-1}FVt} V^{-1} x(0)$$

$$x(t) = e^{Ft} x(0)$$

$$e^{Ft} = V \underbrace{e^{V^{-1}FVt}} e^{Ft} V^{-1}$$

solve for  $x(t)$  with matrix exponential

Who cares?

$$e^{Ft} = V e^{\underbrace{V^{-1} F V t}} V^{-1}$$

this is easy to find if  $V^{-1} F V$  is diagonal  
so let's choose  $V$  so this is true

our goal - find invertible  $V$  such that

$$V^{-1} F V = \text{diag}(s_1, \dots, s_n) \leftarrow \text{this is the same as } FV = V \text{diag}(s_1, \dots, s_n)$$

for example, suppose  $F$  is  $2 \times 2$ :

$$V = [v_1 \quad v_2] \quad \text{diag}(s_1, s_2) = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

↑            ↑  
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columns of  $V$  (both are  $2 \times 1$ )

then:

$$FV = [Fv_1 \quad Fv_2] \quad V \text{diag}(s_1, s_2) = [s_1 v_1 \quad s_2 v_2]$$

$Fv_1 = s_1 v_1$              $Fv_2 = s_2 v_2$

if

the eigenvalues  $s_1, \dots, s_n$  of  $F$  are all distinct  
and we define a matrix

$$V = [v_1 \dots v_n]$$

with the corresponding eigenvectors in each column

then

$$x(t) = e^{F(t-t_0)} x(t_0) = V \underbrace{e^{\text{diag}(s_1, \dots, s_n)(t-t_0)}}_{\text{diag}(e^{s_1(t-t_0)}, \dots, e^{s_n(t-t_0)})} V^{-1} x(t_0)$$

$$\text{diag}(e^{s_1(t-t_0)}, \dots, e^{s_n(t-t_0)})$$

what if  $s = a + jb$  ???

$$e^{(a+jb)(t-t_0)} = e^{a(t-t_0)} e^{jb(t-t_0)}$$

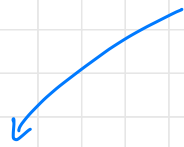
$$= e^{\uparrow a(t-t_0)} (\cos(b(t-t_0)) + j \sin(b(t-t_0)))$$

REAL PART OF EIGENVALUE

The system

$$\dot{x} = Fx$$

$$x(t) \rightarrow 0 \text{ as } t \rightarrow \infty$$



is asymptotically stable if and only if all eigenvalues of  $F$  have negative real part.