AE353 (Spring 2021)

Day 11 – Stability

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nonlinear ODEs

$\downarrow$ put in standard form

$$\dot{m} = f(m,n)$$
$$l = g(m,n)$$

$\downarrow$ linearize

$$x = m - m_e$$
$$u = n - n_e$$
$$y = l - g(m_e, n_e)$$
$$0 = f(m_e, n_e)$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\downarrow$ choose input

$$u = -Kx$$

$\downarrow$ find closed-loop system

$$\dot{x} = (A-BK)x = Fx$$
$$y = (C-DK)x = Gx$$

$\downarrow$ solve by matrix exponential

$$x(t) = e^{F(t-t_0)}x(t_0)$$
$$y(t) = Ce^{F(t-t_0)}x(t_0)$$

for which choices of K does $x(t) \to 0$ as $t \to \infty$???

for which choices of K does $x(t) \to 0$ as $t \to \infty$???

STRATEGY

1. Answer this question in the special case when F is diagonal
2. Show how to rewrite (almost) any F as diagonal
3. Answer this question for (almost) any F

$$\dot{x} = Fx$$

suppose F is diagonal

$$F = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

then:

$$e^{Ft} = I + Ft + \frac{1}{2!}(Ft)^2 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} s_1 t & 0 \\ 0 & s_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(s_1 t)^2 & 0 \\ 0 & \frac{1}{2}(s_2 t)^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + s_1 t + \frac{1}{2}(s_1 t)^2 + \cdots & 0 \\ 0 & 1 + s_2 t + (s_2 t)^2 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{s_1 t} & 0 \\ 0 & e^{s_2 t} \end{bmatrix}$$

← when F is diagonal, e^{Ft} is easy to find coordinate invariance

$z = V^{-1}x$

$\dot{x} = Fx$

$\downarrow$ plug in $x = Vz$ for some invertible V

$V\dot{z} = FVz$

$\downarrow$ solve for $\dot{z}$

$\dot{z} = V^{-1}FVz$

$\downarrow$ solve for $z(t)$ with matrix exponential

$z(t) = e^{V^{-1}FVt} z(0)$

$\downarrow$ plug in $z = V^{-1}x$ and solve for x

$V^{-1}x(t) = e^{V^{-1}FVt} V^{-1}x(0)$

$x(t) = \boxed{V e^{V^{-1}FVt} V^{-1}} \underline{x(0)}$

$x(t) = \boxed{e^{Ft}} \underline{x(0)}$

$$e^{Ft} = V e^{V^{-1}FVt} V^{-1}$$

solve for $x(t)$ with matrix exponential who cares?

$$e^{Ft} = V \underbrace{e^{V^{-1}FVt}}_{} V^{-1}$$

this is easy to find if $V^{-1}FV$ is diagonal
so let's choose V so this is true

our goal – find invertible V such that

$$V^{-1}FV = \mathrm{diag}(s_1, \ldots, s_n)$$ ← this is the same as $FV = V\,\mathrm{diag}(s_1, \ldots, s_n)$

for example, suppose F is 2×2:

$$V = [v_1 \quad v_2] \qquad \mathrm{diag}(s_1, s_2) = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

columns of V (both are 2×1)

then:

$$FV = [Fv_1 \quad Fv_2] \qquad V\,\mathrm{diag}(s_1, s_2) = [s_1 v_1 \quad s_2 v_2]$$

$Fv_1 = s_1 v_1$

$Fv_2 = s_2 v_2$

if

the eigenvalues $s_1, \dots, s_n$ of F are all <u>distinct</u>
and we define a matrix

$$V = [v_1 \ \cdots \ v_n]$$

with the corresponding eigenvectors in each column

then

$$x(t) = e^{F(t-t_0)} x(t_0) = V \underbrace{e^{\mathrm{diag}(s_1, \dots, s_n)(t-t_0)}}_{\mathrm{diag}\left(e^{s_1(t-t_0)}, \dots, e^{s_n(t-t_0)}\right)} V^{-1} x(t_0)$$

what if $s = a + jb$???

$$e^{(a+jb)(t-t_0)} = e^{a(t-t_0)} e^{jb(t-t_0)}$$

$$= e^{a(t-t_0)} \left(\cos\left(b(t-t_0)\right) + j \sin\left(b(t-t_0)\right) \right)$$

REAL PART OF EIGENVALUE

$x(t) \to 0$ as $t \to \infty$

The system

$$\dot{x} = Fx$$

is asymptotically stable if and only if all eigenvalues of F have negative real part.